Three-Frequency Motion and Chaos in the Ginzburg-Landau Equation

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The Ginzburg-Landau equation with periodic boundary conditions on the interval $[0, 2\pi/q]$ is integrated numerically for large times. As q is decreased, the motion in phase space exhibits a sequence of bifurcations from a limit cycle to a two-torus to a three-torus to a chaotic regime. The three-torus is observed for a finite range of q and transition to chaotic flow is preceded by frequency locking.

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The aim of this article is to describe one possible transition to chaos in the long-time solutions of the Ginzburg-Landau equation

$$iA_t + (1 - ic_0)A_{xx} = ic_0 A - (1 + ic_0)|A|^2 A, \qquad (1)$$

where the time variable is t, the spatial coordinate is x, $A(x,t)$ is an unknown complex amplitude, and c_0 is a real parameter. Equations of this type govern the amplitude evolution of instability waves close to marginal stability in fluid systems such as Bénard convection,[1] Taylor-Couette flow,[2] and plane Poiseuille flow.[3] When $c_0 \neq 0$, the equation is dissipative. When $c_0 = 0$, it reduces to the integrable cubic nonlinear Schrödinger equation which governs the modulations of inviscid, deep-water gravity waves.[4] Linear theory[5] reveals that the Stokes solution $A(x,t) = \exp(it)$ of Eq. (1) is unstable to fluctuations of sideband wave number q in the range $0 < q^2 < 2(1 - c_0^2)/(1 + c_0^2)$. In the present study we examine by means of numerical integration the long-time behavior of this instability for initial conditions of the form $A(x,0) = 1 + 0.02 \cos qx$. The parameter c_0 is maintained at a constant value $c_0 = 0.25$ and periodic boundary conditions are imposed on the interval $[0, 2\pi/q]$. The character of the solutions is studied by varying q within the linearly unstable range $0 < q < 1.3$.

A pseudospectral method[6] is used to perform the numerical integration. With a particular choice of initial conditions, the complex amplitude $A(x,t)$ is expanded into the spatial Fourier series

$$A(x,t) = \sum_{n=0}^{N-1} [a_n(t) + ib_n(t)] \cos(nqx), \qquad (2)$$

where $a_n(t)$ and $b_n(t)$ are real and N is the truncation parameter. The inifinite-dimensional system (1) is effectively approximated by a system of finite dimension $2N$, its degrees of freedom being represented in phase space by the functions $a_n(t)$ and $b_n(t)$. The number N was taken to be $N = 64$, the time step was chosen as $\Delta t = 0.001$, and the integration was carried to approximately 134 basic Stokes periods. It was checked that the results are not altered by a change in the value of N, provided N is large enough. Note that, according to our results, only the lowest five spatial Fourier modes have significant energy, which is well below the value chosen for N.

Qualitative changes in the behavior of the phase-space trajectories are followed by making use of power spectra and Poincaré sections. In the range $0.6 < q < 1.31$, the motion is periodic and trajectories are confined to a limit cycle. The power spectrum exhibits only one frequency f_1 and its harmonics. As q is decreased below the value 0.6 and remains in the interval $0.52 < q < 0.6$, the flow becomes quasiperiodic; i.e., its spectrum [Fig. 1(a)] is composed of two independent frequencies f_1 and f_2 and their combination harmonics. The Poincaré section [Fig. 1(b)] is a closed curve, which further suggests that trajec-

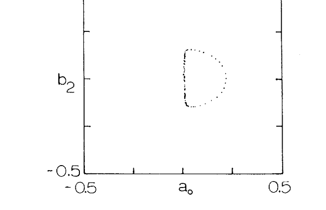

FIG. 1. Two-frequency regime, $q = 0.55$: (a) Power spectrum of $A(\pi/q, t)$; (b) Poincaré section of $A(x,t)$, constructed by plotting b_2 vs a_0 whenever $a_2 = 0$ [see Eq. (2)].

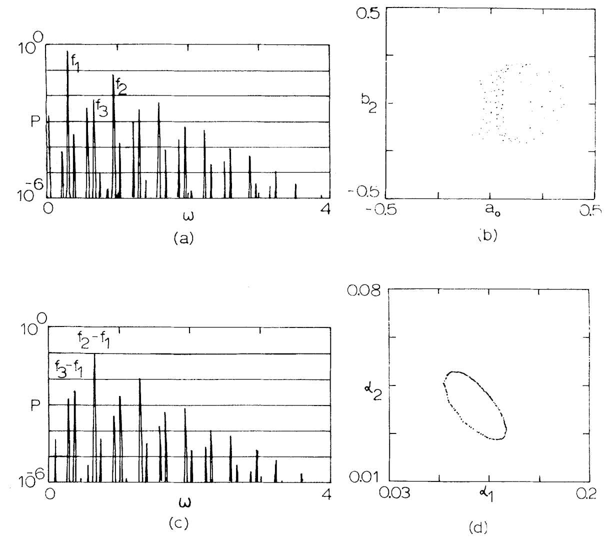

FIG. 2. Three-frequency regime, $q=0.51$: (a) Power spectrum of $A(\pi/q,t)$; (b) Poincaré section of $A(x,t)$ constructed as in Fig. 1(b); (c) power spectrum of $|A|^2(\pi/q,t)$; (d) Poincaré section of $|A|^2(x,t)$ constructed by plotting α_2 vs α_1 whenever $\alpha_0=0.01$ [see Eq. (3)].

tories are attracted to a two-torus. In the range $0.49<q<0.52$, the power spectrum [Fig. 2(a)] stays discrete, but in order to explain the presence of all frequency components in terms of combination harmonics, one must assume that a third frequency f_3 has appeared. Since a three-torus does not reduce to a curve when cut by a two-dimensional plane, the Poincaré section of Fig. 2(b) is consistent with the existence of a three-torus. In order to bring the motion back into three-dimensional Euclidian space, we consider instead the evolution of the square of the modulus $|A|^2(x,t)$ which admits a spatial Fourier decomposition of the form

$$|A|^2(x,t)=\sum_{n=0}^{N-1}\alpha_n(t)\cos nqx. \tag{3}$$

It is then found that the spectrum of $|A|^2(x,t)$ [Fig. 2(c)] only consists of two independent frequencies f_2-f_1 and f_3-f_1, with an additional peak at zero frequency. By comparison with Fig. 2(a), the origin of the frequency axis has been shifted by an amount f_1. A Poincaré section [Fig. 2(d)] taken in the phase space of the functions $\alpha_n(t)$ confirms that the asymptotic trajectories of $|A|^2(x,t)$ are located on a two-torus. In this range of sideband wave numbers, the evolution of $A(x,t)$ is, therefore, characterized by three independent frequencies, one of them (f_1) being associated with

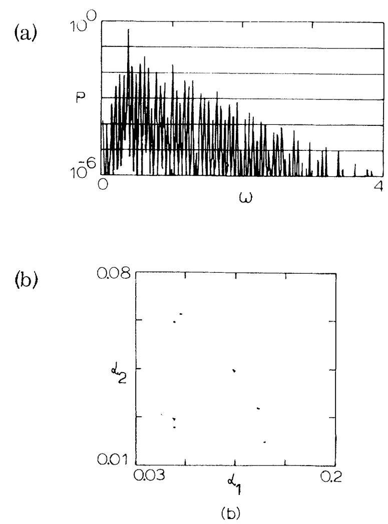

FIG. 3. Frequency locking, $q=0.49$: (a) Power spectrum of $A(\pi/q,t)$; (b) Poincaré section of $|A|^2(x,t)$ constructed as in Fig. 2(d).

the phase of $A(x,t)$ and the other two with the amplitude $|A|(x,t)$. The spectrum of Fig. 3(a) and the Poincaré section of Fig. 3(b) suggest that, at $q=0.49$, the motion of $|A|(x,t)$ abruptly shrinks back to a limit cycle. Correspondingly, the motion of $A(x,t)$ is reduced to a two-torus. This state is then immediately followed by the appearance of a broad spectrum [Fig. 4(a)] and a scattered Poincaré section [Fig. 4(b)], which are both indicative of a chaotic regime.

In summary, we conclude that the motion of A

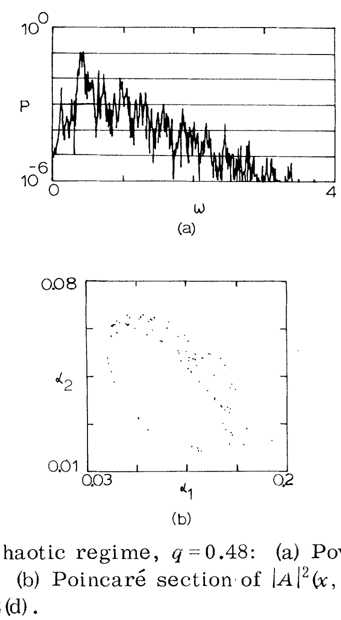

FIG. 4. Chaotic regime, $q=0.48$: (a) Power spectrum of $A(\pi/q,t)$; (b) Poincaré section of $|A|^2(x,t)$ constructed as in Fig. 2(d).

($|A|$) has undergone successive bifurcations from a limit cycle (fixed point) to a two-torus (limit-cycle) to a three-torus (two-torus) to chaos. As just mentioned, the last bifurcation may involve a preliminary frequency locking. The occurrence of a quasiperiodic motion with three incommensurate frequencies just prior to the onset of a chaotic regime has previously been observed in Bénard convection experiments.[7] To our knowledge, this is the first instance in which such a sequence of bifurcations arises in numerical simulations of partial differential equations. According to Newhouse, Ruelle, and Takens,[8] perturbations of a three-torus can produce strange axiom-A attractors. The present results suggest that a three-frequency motion can actually be stable for a finite interval of values of the control parameter q.

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[1] A. C. Newell and J. A. Whitehead, J. Fluid Mech. 38, 279 (1969).
[2] S. Kogelman and R. C. DiPrima, Phys. Fluids 13, 1 (1970).
[3] K. Stewartson and J. T. Stuart, J. Fluid Mech. 48, 529 (1971).
[4] H. C. Yuen and B. M. Lake, Ann. Rev. Fluid Mech. 12, 303 (1980).
[5] J. T. Stuart and R. C. DiPrima, Proc. Roy. Soc. London, Sect. A 362, 27 (1978).
[6] B. Fornberg and G. B. Whitham, Philos. Trans. Roy. Soc. London 289, 373 (1978).
[7] J. P. Gollub and S. V. Benson, J. Fluid Mech. 100, 449 (1980).
[8] S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. 64, 35 (1978).

Sequence of Instabilities in Electromagnetically Driven Flows between Conducting Cylinders

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An experiment on electromagnetically driven flows shows sequences of instabilities involving overstability and slow oscillations of the cellular structure before the onset of turbulence.

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Transition to turbulence in hydromagnetic Taylor-vortex flows of liquid metal has not received much attention until now. This situation contrasts with the hydrodynamic case (no magnetic field) for which extensive experimental data exist.[1] On physical grounds the effect of an imposed external magnetic field on cellular flows is expected to alter profoundly the sequence of events leading to turbulence. The present Letter is related to a limited experimental investigation of Taylor instability subjected to an external magnetic field.

Figure 1 shows the experimental arrangement. The mercury is confined between two "long" copper cylinders, 3.84 cm in height, and 8.0 and 8.22 cm in diameter, respectively; a thin layer of nickel (25 μm) and gold (2–3 μm) has been deposited on the copper surfaces so that the electrical and mechanical conditions of the flow are well defined on the boundaries. The duct is axially

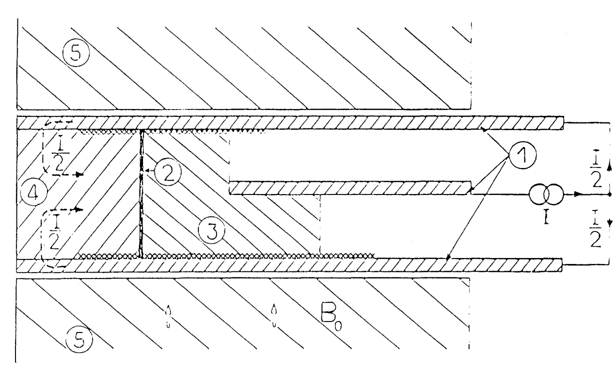

FIG. 1. The experimental arrangement. 1, alloy aluminum plates; 2, mercury; 3, outer copper cylinder; 4, inner copper cylinders; 5, electromagnet.