Two-Mode Structure of Alfvén Surface Waves

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The Alfvén surface waves propagating along a viscous conducting fluid–vacuum interface have been studied. It is found that besides the "ordinary" Alfvén surface waves, modified by viscosity effects, the interface can support a second mode which is the overdamped solution of the dispersion equation. The possibility of observation of a two-mode structure of Alfvén surface waves in the laboratory and in the solar coronal plasmas is discussed.

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In this Letter we study the viscosity effects on Alfvén surface waves (ASW) propagating along a plane interface between vacuum and a conducting viscous medium. We find that the interface can support two qualitatively distinct modes when the parameter V ($V \equiv \nu_1 \omega/\rho_{01} v_{A1}^2$) is greater than a critical value $V_c(\alpha)$, where ν_1, ρ_{01}, and v_{A1} are respectively the coefficient of viscosity, density, and bulk Alfvén speed in the conducting medium 1, and $\alpha = B_{02}/B_{01}$ is the interface parameter measuring the discontinuity of the magnetic field strength across the interface. One of the modes is the "ordinary" ASW modified by viscosity effects, but the second mode is the overdamped solution of the dispersion equation arising due to the presence of a magnetic field along the surface of a viscous conducting fluid. The two-mode structure of ASW is very similar to the two-mode structure of capillary surface waves arising along the liquid surface in the presence of surface tension and shear viscosity.[1,2] This similarity could be due to the fact that the rigidity induced by the magnetic field in the surface of the conducting fluid acts like surface tension.

We discuss the dispersion relation by considering the frequency to be real and the wave vector complex with a hope that the two modes of ASW can be observed in the laboratory by use of the spatial methods of experimental technique.[1] Moreover, the imaginary part of the propagation constant gives us an estimate of the distance over which the "ordinary" ASW is damped by viscosity. These results can have relevance to the study of ASW in the heating of solar coronal plasma, where it has been noted[3] recently that in the process of resonant absorption of ASW viscosity dominates over the kinetic effects.

In the magnetohydrodynamic approximation the linearized equations governing the electromagnetic and hydrodynamic properties of an incompressible viscous fluid of mass density ρ_0 embedded in an external magnetic field $\vec{B}_0$ are

$$\nabla \cdot \vec{v} = 0, \tag{1}$$

$$\rho_0 \partial \vec{v}/\partial t = -\nabla p + (4\pi)^{-1}(\nabla \times \vec{b}) \times \vec{B}_0 + \nu \nabla^2 \vec{v}, \tag{2}$$

$$\partial \vec{b}/\partial t = \nabla \times (\vec{v} \times \vec{B}_0), \tag{3}$$

$$\nabla \cdot \vec{b} = 0, \tag{4}$$

where ν is the coefficient of viscosity, $\vec{v}$, p, and $\vec{b}$ are the perturbed fluid velocity, pressure, and magnetic field, respectively.

For small perturbations of the form $f(x,y,z,t) \equiv f(x) \exp i(ly + kz - \omega t)$ and $\vec{B}_0 = B_0 \hat{z}$, Eqs. (1)–(4) can be reduced to give the system of differential equations in v_x and v_z as

$$kD(D^2 - \tau^2)v_x + iK^2(D^2 - \tau^2)v_z = 0, \tag{5}$$

$$k(D^2 - \tau^2)v_x + iD(D^2 - \tau^2)v_z = 0, \tag{6}$$

where $D = d/dx$,

$$\tau^2 = K^2 - (i\rho_0/\nu\omega)(\omega^2 - k^2 v_A^2),$$

$K^2 = k^2 + l^2$, and v_A is the Alfvén wave speed. Equations (5) and (6) can be reduced to give

$$(D^2 - \tau^2)(D^2 - K^2)v_x = 0. \tag{7}$$

Consider two fluid media 1 and 2 filling the half spaces $x > 0$ and $x < 0$, respectively. Solving Eq. (7) we get

$$v_{x1} = A_1 e^{-\tau_1 x} + A_2 e^{-Kx}, \quad x > 0, \tag{8}$$

$$v_{x2} = A_3 e^{\tau_2 x} + A_4 e^{Kx}, \quad x < 0, \tag{9}$$

where the A's are arbitrary constants. Substituting Eqs. (8) and (9) in Eq. (5) and solving, we get

$$v_{z1} = E_1 e^{-\tau_1 x} - (iA_2 k/K)e^{-Kx}, \quad x > 0, \tag{10}$$

$$v_{z2} = E_2 e^{\tau_2 x} + (iA_4 k/K)e^{Kx}, \quad x < 0, \tag{11}$$

where the E's are arbitrary functions of the A's.

To represent surface waves we also impose the conditions

$$\mathrm{Re}(\tau_{1,2})>0, \quad \mathrm{Re}(K)>0. \tag{12}$$

By use of results (8)–(11) in Eqs. (1)–(4) the field components in media 1 and 2 can be calculated. The dispersion relation is obtained by applying the four boundary conditions that (i) normal velocity, (ii) tangential velocity, (iii) tangential viscous stress, and (iv) total pressure are continuous across the boundary. It is to be mentioned that in calculation of the total pressure on each side of the boundary, the coefficients of E_1 and E_2 vanish and the remaining three conditions (i)–(iii) can be expressed in terms of v_{x1} and v_{x2}. Matching the boundary conditions at $x=0$, we get the dispersion relation as

$$i[\rho_{01}(\omega^2-k^2v_{A1}^2)+\rho_{02}(\omega^2-k^2v_{A2}^2)][\rho_{01}(\omega^2-k^2v_{A1}^2)(\tau_2-K)+\rho_{02}(\omega^2-k^2v_{A2}^2)(\tau_1-K)]$$
$$+4iK\rho_{01}(\omega^2-k^2v_{A1}^2)\rho_{02}(\omega^2-k^2v_{A2}^2)-4K^2\omega(\nu_1-\nu_2)[\rho_{01}(\omega^2-k^2v_{A1}^2)(\tau_2-K)-\rho_{02}(\omega^2-k^2v_{A2}^2)(\tau_2-K)]$$
$$+i4K^3\omega^2(\nu_1-\nu_2)^2(\tau_1-K)(\tau_2-K)=0. \tag{13}$$

The structure of Eq. (13) suggests that l can be taken to be zero without effecting the qualitative nature of the results. When we consider the plasma-vacuum interface, Eq. (13) with $l=\rho_{02}=\nu_2=0$ then reduces to

$$(\omega^2-k^2v_{A1}^2)^2-\frac{B_{02}^2k^2}{4\pi\rho_{01}}(\omega^2-k^2v_{A1}^2)+\frac{i4\nu_1k^2\omega}{\rho_{01}}(\omega^2-k^2v_{A1}^2)+\frac{4\nu_1^2\omega^2k^4}{\rho_{01}^2}\left(\frac{\tau_1}{k-1}\right)=0. \tag{14}$$

It is interesting to see that when the magnetic field is equal to zero, Eq. (14) reduces to

$$1+(i4k^2\nu_1/\omega\rho_{01})+4(k^4\nu_1^2/\omega^2\rho_{01}^2)(\tau_1/k-1)=0, \tag{15}$$

which is the dispersion relation for the surface waves at the viscous-fluid–vacuum interface when surface tension is zero.[1]

We also note that in the limit $\nu_1\to 0$ Eq. (14) gives

$$\omega/k=[(B_{01}^2+B_{02}^2)/(4\pi\rho_{01})]^{1/2}\equiv v_{AS}, \tag{16}$$

the phase velocity of the ASW at the plasma-vacuum interface.[4]

Normalizing Eq. (14) we get

$$(x^2-1)^2-\alpha^2(x^2-1)+i4v(x^2-1)+4v^2T_1-4v^2=0, \tag{17}$$

where

$$x=\frac{\omega}{kv_{A1}}, \quad V=\frac{\nu_1\omega}{\rho_{01}v_{A1}^2}, \quad T_1=\frac{\tau_1}{k}=\left(1-\frac{i(x^2-1)}{V}\right)^{1/2}.$$

Squaring the irrational equation (17) after bringing the term with T_1 to the right-hand side and taking out a common factor (x^2-1) which represents a bulk mode, we get

$$x^6+x^4(-3-2\alpha^2+8iV)+x^2\{(1+\alpha^2)[(1+\alpha^2)+2(1-4iV)]-8V(i+3V)\}$$
$$+(1+\alpha^2)[-1-\alpha^2+8V(i+V)]+16V^2(1-iV)=0. \tag{18}$$

Equation (18), with the substitution $y=x^2$, reduces to a cubic equation in y, the roots of which are given by the usual formulas.[5] Taking $x=\pm\sqrt{y}$, we get all the six roots of Eq. (18). To avoid spurious roots due to squaring we take only those roots which satisfy Eq. (17). Further, to represent a surface mode, the roots should also satisfy the condition $\mathrm{Re}(\tau_1)>0$, $\mathrm{Re}(k)>0$.

Taking $k=k_r+ik_i$, the values of k_r and k_i can be obtained from the calculated values of x. The values thus obtained in units of ω/v_{A1} are shown in Figs. 1(a), 1(b), and 1(c) as a function of V for $\alpha^2=0.2$, 1.0, and 1.5, respectively. These figures show that there is only one surface-wave mode which is the ASW modified by viscosity for low values of V. As V increases a second mode appears at a critical value of V, say $V_c(\alpha)$. $V_c(\alpha)$ is an increasing function of α and approaches the overdamped solution, for which $k_i>k_r$, at $V=V_0$. Thus for values of $V<V_c$ there is only one ordinary ASW mode which exists even in the absence of viscosity as the pure ASW mode, for the region $V_c<V<V_0$ there are two damped propagating

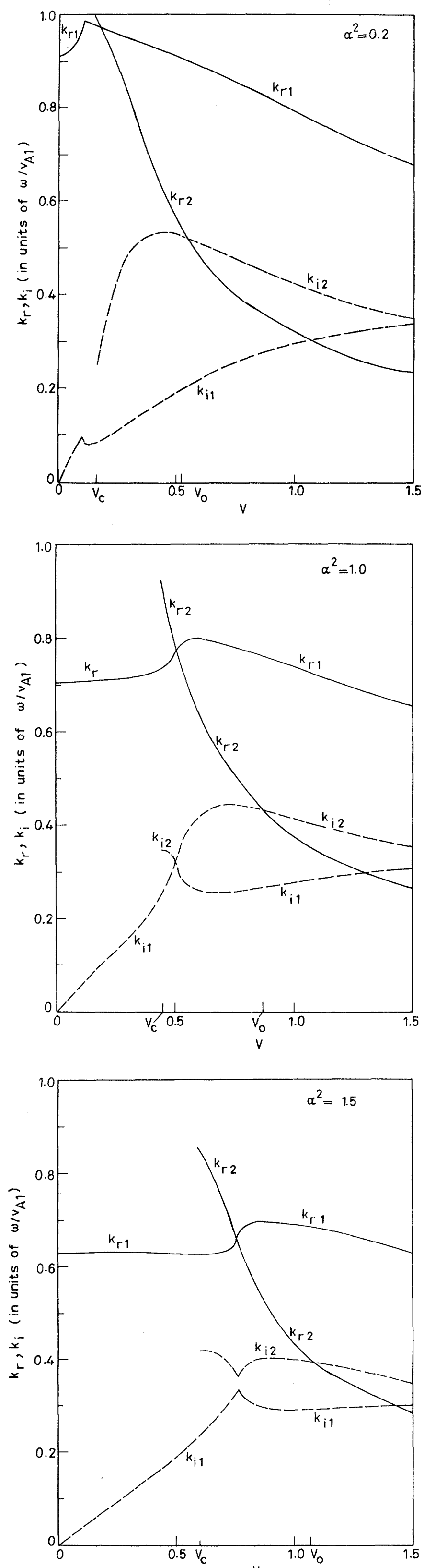

FIG. 1. Variation of k_r and k_i with the parameter V for (a) $\alpha^2 = 0.2$, (b) $\alpha^2 = 1.0$, (c) $\alpha^2 = 1.5$.

modes, and for $V > V_0$ there is one damped propagating mode and one overdamped mode. Other interesting features are that for the values of the parameter $V \approx V_c$ both the propagating and damping constants of ordinary ASW show a sharp maximum peak and for low values of the interface parameter α, the value of V_c at which the second mode sets in can become very small.

In the limit $V \ll 1$, Eq. (17), on neglect of terms of order higher than V, becomes

$$x^2 = 1 + \alpha^2 - 4iV,$$

which on assumption $k_i \ll k_r$ gives

$$k_i \simeq \frac{2V}{(1+\alpha^2)^{3/2}} \frac{\omega}{v_{A1}}, \qquad k_r \simeq \frac{1}{(1+\alpha^2)^{1/2}} \frac{\omega}{v_{A1}} = \frac{\omega}{v_{AS}}. \tag{19}$$

The approximate solutions in Eq. (19) are correct for values of $\alpha \geq 1$. Figure 1(b) and 1(c) show k_r as a constant and k_i as a linear function of V for values of $V < 0.3$ and 0.5, respectively. Figure 1(a), however, shows that the curves for k_i and k_r deviate from Eq. (19) even when $V < 0.1$. A criterion for propagation of ASW, deduced from Eq. (19) by taking $k_i < k_r$, is given as

$$\nu_1 \omega / \rho_{01} < (B_{01}{}^2 + B_{02}{}^2)/8\pi\rho_{01}. \tag{20}$$

An estimate of the relevant range of V to study the two-mode structure of ASW in the laboratory can be made by considering a highly viscous and conducting liquid, say mercury. The value of V for a frequency of 800 Hz ($\omega = 5 \times 10^3$/sec) in a magnetic field B_0 is given by $V = (975\ \mathrm{G}^2)/B_0{}^2$. Hence in a magnetic field of 500 G, $V = 3.9 \times 10^{-3}$. From Fig. 1(b) for $\alpha^2 = 1.0$, we find that for this value there can be only one mode. However, since V is inversely proportional to the square of the magnetic field, by lowering the magnetic field to say 40 G, we find $V = 0.61$. For this value, the second mode can very well be observed. Recently Sohl, Miyano, and Ketterson[6] have developed a technique to study capillary-wave propagation. The wide dynamic range of their technique allows them to observe highly damped waves on viscous fluids conveniently. The two-mode structure of ASW can be observed by use of a similar technique.

When we consider the situations in coronal loops, the field-aligned viscosity[7,8] is $\nu/\rho = 6.9 \times 10^7 T^{5/2}/n$ cm^2/sec, where T is the temperature and n is the particle density. Hence for the temperatures of the order 10^5–10^6 K with a particle density of $10^9/\mathrm{cm}^3$, $\nu/\rho = 2.181\times10^{11}-6.9\times10^{13}$ $\mathrm{cm}^2/\mathrm{sec}$. For $v_{A1} = 10^8$ cm/sec and $\omega = 1/\mathrm{sec}$, $V = 2.18\times10^{-5}-6.9\times10^{-3}$. Considering the fact that V is very low, the probability that the second mode of ASW can exist along the coronal loops is less. However, our results assume $\rho_{02}/\rho_{01} \equiv \eta = 0$, whereas for coronal loops η is[3] in the range 2 – 3 and α is small. Since V_c is a sensitive function of the interface parameter α, a further study of functional dependence of V_c on η is necessary to arrive at any definite conclusion.

Finally, we make an estimate of the damping of ordinary ASW in coronal loops. If damping is very rapid, the ASW absorption may not be a possibility. For $V \approx 0.06$, from Fig. 1(a), we have $k_{r1} = 0.935\omega/v_{A1}$, $k_{i1} = 0.065\omega/v_{A1}$, which gives $\lambda_{r1} = 2\pi/k_{r-1} = 6.7\times10^3$ km, $\lambda_{i1} = 9.67\times10^4$ km, for $v_{A1} = 10^8$ cm/sec and $\omega \approx 1/\mathrm{sec}$. Thus ordinary ASW can propagate to a distance of $\simeq 9.67\times10^4$ km which is 14 times the propagation wavelength before being damped.

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