suming evaluation of higher-order electron-electron interaction effects. Thus, if, in the future, the error ranges in the experimental results for M_1 and δ can be reduced substantially below the present ranges,[3,7] combined theoretical and experimental investigations of δ have the potential of providing an accurate choice of $\sin^2\theta_W$. There seems to be significant hope for this since the recent experimental investigation[3] of δ in thallium has reduced the experimental error range by almost 60% as compared with earlier measurements.[14]

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Bistable Limit Cycles in a Model for a Laser with a Saturable Absorber

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Sufficiently long population decay times and sufficiently short dipole decay times in a single-mode model for a laser with saturable absorber permit coexistence of soft-excited oscillations and Q switching (hard-mode sustained relaxation oscillations).

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Systems driven away from equilibrium can exhibit spatial and/or temporal patterns which are *dissipative structures*,[1] thus leading to *synergetic* behavior.[2] The new states can be induced either *softly* as in a second-order, continuous phase transition or through *hard excitation* as in a first-order, discontinuous case. For the latter possibility where the transition is a consequence of a finite-amplitude disturbance we speak of subcritical instability or of transition with metastability. A specific case of nonequilibrium transitions is the appearance of multiple steady states which in laser physics permits optical *bistability*.[3] Here we present evidence of multiplicity of oscillatory states with coexistence of soft- and hard-induced limit cycles in a laser with a saturable absorber.

In a recent Letter[4] evidence was given for the onset of Q switching[5,6] in a model for a single-mode laser with a saturable absorber.[3] Such a limit cycle appears as a hard-mode sustained relaxation oscillation for sufficiently long population decay times and sufficiently short dipole decay times. The model considered in Ref. 4 did not account for the phases in the electric and polarization fields. However, because of the chosen range of parameter values the phases were not expected to play a significant role. In the present Letter we report the results found for a range of parameter values where a nontrivial role may be played by the phases as has been already emphasized by several authors.[7,8] Figure 1 illustrates the expected behavior as predicted by linear stability analysis of the nonlasing steady state in the model discussed in Ref. 3 or 4.

The problem refers to the following dimensionless equations:

$$\dot{a}_r = \rho[-a_r + Ap_r + r_1(1-C)\bar{p}_r], \quad (1a)$$

$$\dot{a}_i = \rho[-a_i + Ap_i + r_1(1-C)\bar{p}_i], \quad (1b)$$

$$\dot{p}_r = a_r(1-d) - p_r, \quad (1c)$$

$$\dot{p}_i = a_i(1-d) - p_i, \quad (1d)$$

$$\dot{\bar{p}}_r = a_r(1-\bar{d}) - r_1\bar{p}_r, \quad (1e)$$

$$\dot{\bar{p}}_i = a_i(1-\bar{d}) - r_1\bar{p}_i, \quad (1f)$$

$$\dot{d} = \omega(-d + a_r p_r + a_i p_i), \quad (1g)$$

$$\dot{\bar{d}} = \omega(-r_2\bar{d} + a_r\bar{p}_r + a_i\bar{p}_i). \quad (1h)$$

a_r and a_i denote the real and imaginary parts of the dimensionless electric field. p_r and p_i account for the polarization of the active atoms. $d-1$ is the normalized emitter's atomic inversion. Barred quantities refer to the absorber.

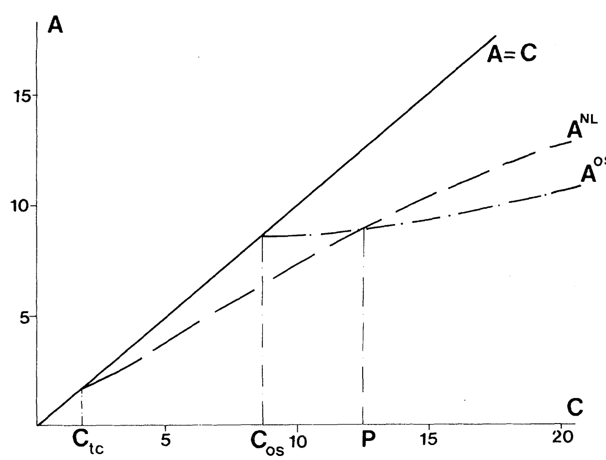

FIG. 1. Boundaries of stability of the emissionless steady state of (1) for $\rho = 0.1$, $\omega = 0.01$, $r_1 = 0.4$, and $r_2 = 1$. $A = C$ is the line of exchange of stabilities (linear theory). A^{NL} is the line of hard transition from the emissionless state to the steady lasing state. A^{os} is the line of overstability (linear theory) where for $A > A^{os}$ and $C > C_{os}$ we expect an oscillatory lasing state. Past P, to the right, there is coexistence of soft- and hard-mode excited oscillatory states.

ρ is the ratio of the photon decay rate in the cavity to the dipole decay rate. r_1 is the ratio of the two dipole decay rates (absorber to emitter). r_2 is the ratio of the two population decay rates (absorber to emitter) and ω is the ratio of the two decay rates of the emitter (population to dipole). A and C are essentially the pumping rates of the emitter and absorber, respectively.

As in Ref. 4, we set $r_2 = 1$, i.e., we take the population decay time or longitudinal relaxation time, T_1, the same for both active and passive atoms, which is consistent with the assumption of resonance between the emitting and absorbing transitions. We fix $\omega = 0.01$, i.e., we take the dipole decay time or transverse relaxation time, T_2, to be two orders of magnitude smaller than T_1. We also take $\rho = 0.1$ and thus $\omega < \rho$. Note that Eqs. (1) provide nontrivial phase effects, i.e., nonvanishing values of a_i and p_i, only when the atomic system is prepared in a coherent superposition state at the initial time.

The steady solutions of (1) are either the emissionless state $a_r = a_i = 0$ or a nonlinear steady emission state with $a_r^2 + a_i^2 = X^2 \neq 0$, where X is any of the positive roots of $X^4 + X^2(1 - A + r_1C) + r_1(CA) = 0$. Note that we only have nonvanishing positive roots for $C > C_{tc} \equiv (1-r_1)^{-1}$ and that below a certain value of A, called A^{NL}, there is only the solution $X = 0$. Thus A^{NL} (see Fig. 1) corresponds to the appearance of four solutions in the algebraic equation. We also have $d = X^2/(1+X^2)$, $\bar{d} = X^2/(r_1 + X^2)$, $p_r = a_r/(1+X^2)$, $p_i = a_i/(1+X^2)$, $\bar{p}_r = a_r/(r_1+X^2)$, and $\bar{p}_i = a_i/(r_1+X^2)$.

The emissionless state is unstable to infinitesimal disturbances when

$$A \geq C \quad (2a)$$

or

$$A \geq (\rho + r_1)\frac{1 + r_1 + \rho(1 + r_1C)}{\rho(\rho+1)} \equiv A^{os}. \quad (2b)$$

Along $A = C$ there is *exchange of stabilities* and a transition to the steady lasing state ($X^2 \neq 0$) is expected. This is a soft transition for $C < C_{tc}$, whereas there is a finite-amplitude instability, i.e., a hard transition from $X = 0$ to $X \neq 0$, at $A = A^{NL}$ for $C > C_{tc}$. At $C = C_{os} \equiv (\rho + r_1)/\rho(1 - r_1)$, where the two equalities (2a) and (2b) hold, and all along $A = A^{os}$ there is *overstability*, i.e., a soft Hopf bifurcation with a pair of complex semisimple eigenvalues both of multiplicity two. Their imaginary part is μ_0, where

$$\mu_0^2 = r_1[\rho C(1 - r_1) - (\rho + r_1)]/(1 + \rho).$$

We here restrict consideration to the region $A^{os} < A^{NL}$ (see Fig. 1).

At $A \geqslant A^{os}$ two *stable* limit cycles bifurcate from the emissionless state. We have constructed these two coexisting limit cycles of (1) using a method due to Kielhöfer.[9] One of the limit cycles (LC1) has constant phase and corresponds to the solution reported some time ago by Antoranz and Velarde.[10,11] Thus LC1 is the same whether or not we consider the phases. The other cycle (LC2), already found by Dembinski *et al.*,[8] has linearly growing phase and does not appear in the system of five equations studied by Antoranz, Gea, and Velarde.[4] That LC1 and LC2 both branch *stably* has been verified analytically by means of Floquet's theory[12] and numerically by using the Poincaré map. Thus our results complete the picture recently sketched by Erneux and Mandel.[13]

Figure 2 depicts the results found for some illustrative values of the parameters. With the Poincaré map we have been able to locate the points where the two limit cycles become unstable. Curiously enough both LC1 and LC2 bifurcate to *unstable* tori (Ω_1 and Ω_2, respectively). For LC2 this has been established by means of the time derivative expansion (singular perturbation) procedure.[12,14] As we know[13] the coordinates in the Poincaré map of the corresponding fixed point, we have constructed the bifurcated limit cycle when the fixed point becomes unstable. This limit cycle branches to the wrong side, i.e., bifurcates unstably, in agreement with the numerical evidence obtained by direct computer integration of (1). The latter refers to the observed jump from the fixed point to an outwardly spiraling orbit in the Poincaré map. A similar behavior appears in the Poincaré map of LC1 although for LC1 we have not been able to establish this property analytically.

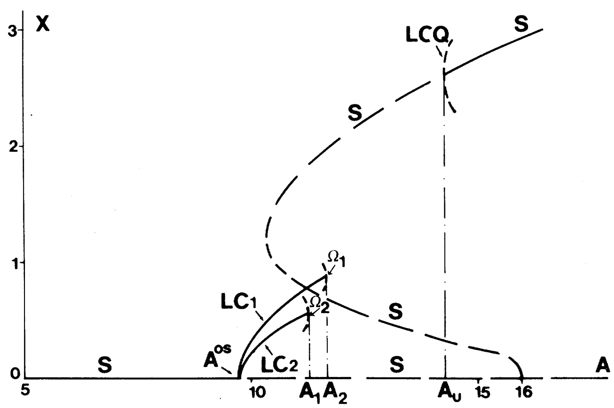

FIG. 2. Bifurcation diagram of (1) at $\rho = 0.1$, $\omega = 0.01$, $r_1 = 0.4$, $r_2 = 1$, and $C = 16$ (see Fig. 1). Solid and broken lines indicate stable and unstable solutions, respectively. S denotes steady states. LCQ bifurcates subcritically at $A = A_u = 14.2823$. LC1 and LC2 bifurcate softly from the emissionless state at $A = A^{os} = 9.7272$. For $A^{os} \leqslant A < A_1 = 11.3466$ there is bistability of softly excited oscillations. For $A_1 < A < A_2 = 11.72$ there is bistability of a softly excited oscillation and the hard excited mode (Q switching). The Q switching is the only available oscillatory state in the region $A_2 < A < A_u$.

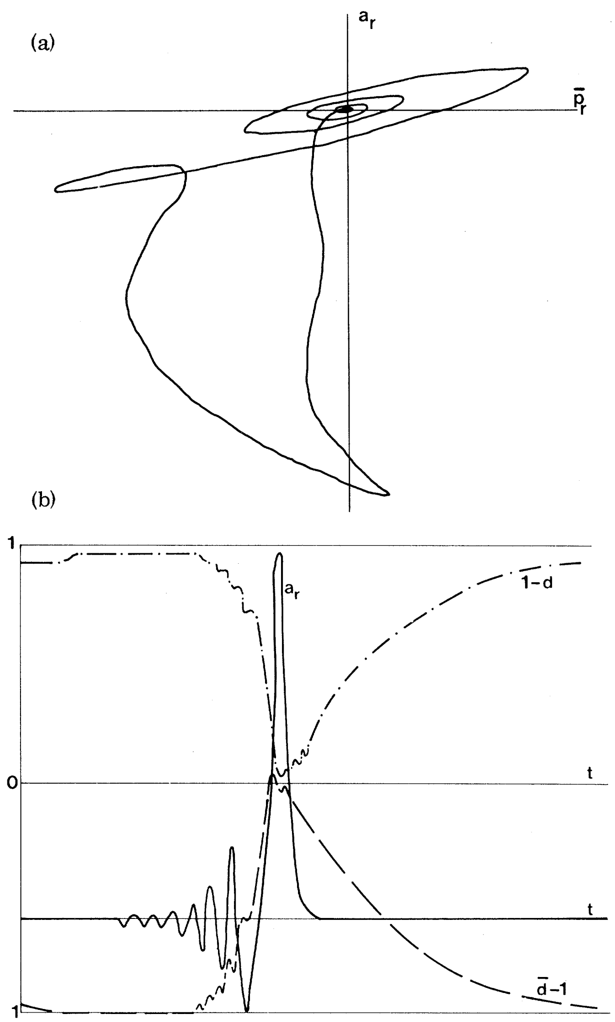

FIG. 3. (a) The Q switching in the phase space (electric field vs polarization). (b) Time evolution, during a period, of population inversions (emitter, $1-d$; absorber, $\bar{d}-1$). For illustration, the solid line accounts for the pulse (not to scale here). Values of 1 (respectively, -1) account for all atoms in the excited (respectively, ground) state. Units are arbitrary.

TABLE I. Characteristics of the Q-switching oscillation (pulse intensity, time period) for $\rho = 0.1$, $\omega = 0.01$, $r_1 = 0.4$, $r_2 = 1$, and $C = 16$. Units are in accordance with the dimensionalization used to obtain (1).

A	Period of oscillation	Pulse intensity (maximum)
11.346	605	39
11.40	591	40
11.5	550	42
12	445	52
12.5	368	61
13	327	70
13.5	272	76
14	234	79

As in Ref. 4 we have also studied the stability of the nonlinear steady lasing state. Here, however, due consideration is given to the phases. With (1) we again find that the upper nonlinear steady state ($X \neq 0$) is *unstable* for $A \leq A_u$ and *stable* past A_u, where the actual value of A_u depends on the parameters in the problem. This value, A_u, is *exactly* the same as the value found with the system of equations used in Ref. 4. As the problem discussed in Ref. 4 is the straightforward reduction of (1) when the phases are disregarded, the latter play no relevant role in the stability analysis of the nonlinear finite-amplitude steady state. They play, however, an interesting role in the evolution of the emissionless state as they permit the appearance of LC2.

Figures 3(a) and 3(b) depict the Q switching found for illustrative values of the parameters. Table I accounts for numerical estimates of the period and intensity of the current at different values of the pumping rate A. A genuine property of this sustained relaxation oscillation is that the pulse peak intensity *increases* with increasing pumping rate which is the opposite behavior to the result found in Ref. 4 where $C < C^{os}$. There the Q switching was the only available oscillatory state of the system. Note also that at $\bar{d} = 1$ the absorber becomes transparent with equal numbers of atoms in the excited and ground states. It actually becomes *active* ($\bar{d} > 1$) for a short interval during the rising of the pulse, and cooperates with the emitter. Moreover this transition is achieved through an oscillatory transient and the maximum is reached before the minimum in the emitter's curve is attained.

In conclusion, the single-mode model[3] for a laser with a saturable absorber permits the coexistence of limit-cycle behavior not only between two *softly* excited oscillations (LC1 and LC2 in the region $A^{os} \leq A < A_1$ of Fig. 2) but also between soft- and hard-excited oscillations (LC1 and LCQ in the region $A_1 < A < A_2$). At $A \lesssim A_1$, LCQ disappears while at $A \gtrsim A_1$, LC2 disappears to yield the *unstable* torus Ω_2. Thus we expect that this torus emanating from LC2 dies at LCQ. The coexistence of LC1 and LCQ is a common feature to (1) and the problem without the phases.[4]

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