Generation of Superfluid Turbulence Deduced from Simple Dynamical Rules

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It is postulated that a quantized-vortex tangle obeys classical vortex dynamics, and that internal line-line crossings can result in topology-changing reconnections. Implementation of these rules leads to a quantitatively successful description of homogeneous superfluid turbulence.

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If superfluid ^{4}He is made to flow rapidly through a channel, it undergoes a transition to a turbulent state in which the fluid is permeated by a dense random tangle of quantized vortex filaments. Although much experimental effort has been devoted to this interesting problem,[1] theoretical work has been limited.[2,3] The most recent attempt[3] to develop a satisfactory description is based on the idea that the action of the normal fluid, which tends to make certain vortex loops grow without limit, competes with the effect of line-line crossings, which tend to keep the tangle random. The resulting order-of-magnitude theory is unsatisfactory in several respects, but since even this treatment is already very complicated, prospects for further progress have seemed dim. It therefore comes as a surprise that, as shown here, a slightly refined version of the ideas presented earlier can be implemented to yield a conceptually satisfactory and quantitatively accurate description of homogeneous superfluid turbulence.

The configuration of the random curves which make up a vortex tangle can be given in the parametric form $\vec{s}=\vec{s}(\xi,t)$, where ξ is the arc length and t is the time. The instantaneous motion of the vortex singularity with respect to the local average superfluid velocity is then given by[3]

$$\partial\vec{s}/\partial t=\beta\vec{s}'\times\vec{s}''+\alpha\vec{s}'\times(\vec{v}_n-\vec{v}_s-\beta\vec{s}'\times\vec{s}''). \qquad (1)$$

Here $\vec{s}'$ is the vector tangent and $\vec{s}''$ the vector curvature of the vortex filament at the point in question; $\vec{v}_n$ and $\vec{v}_s$ are the local average normal and superfluid velocity fields; α is a friction constant related to the conventional Hall-Vinen[4] coefficient B by $\alpha=\rho_n B/2\rho$; and $\beta=(\kappa/4\pi)\ln(c_1/s''a_0)$, where κ is the quantum of circulation, $a_0\sim10^{-8}$ cm is the vortex cutoff parameter, and c_1 is a constant of order 1. Equation (1) is based on the "localized self-induction" approximation,[5] which neglects the long-range effects of the vortex velocity fields. Although this usually introduces only minor errors of order 10%, a special situation arises when lines within the tangle attempt to cross [Fig. 1(a)]. The interaction between the two lines will generate severe local distortions, but these are of minor importance since they are quickly damped out. The important point is that such a crossing raises the possibility that the two lines will suffer a reconnection [Fig. 1(b)]. Such an event changes the topology of the vortex tangle and thus affects its entire future development. I now make the explicit assumption that such reconnections will occur with a probability of order unity whenever two lines try to cross. This simple idea, together with Eq. (1), forms the sole basis of our discussion.

I will first discuss an elementary but important conceptual point. Equation (1) predicts that loops in the vortex tangle, driven by $\vec{v}_{ns}=\vec{v}_n-\vec{v}_s$, will either grow or decay, depending on their size. It has never been clear how a self-sustaining turbulent state can in fact be possible under these conditions. To see the difficulty, consider the highly schematic representation of Fig. 1(c), where the tangle is pictured as consisting of vortex rings.

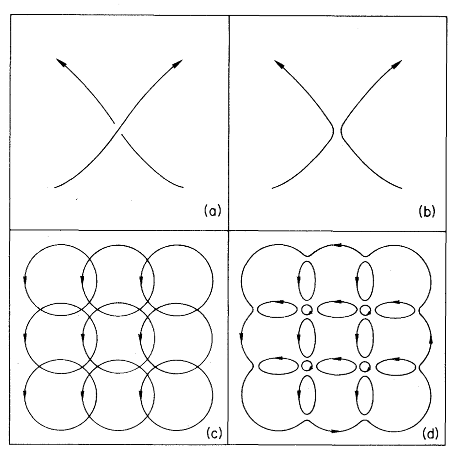

FIG. 1. Effect of topology-changing reconnections. The driving velocity $\vec{v}_{ns}$ points out of the plane of the figure.

It is not hard to make the rings grow by making $\vec{v}_{ns}$ large enough, but in that case they will eventually annihilate at the boundaries leaving the system empty. Calculations involving more realistic descriptions of the vortex tangle lead to the same result. This difficulty is eliminated if the loops are allowed to reconnect as they grow. As can be seen from Fig. 1(d), numerous new small loops can then be formed continually, some of which in turn will grow, maintaining the local steady state. The whole process is accompanied by the continuous creation of outwardly propagating giant loops, which eventually annihilate at the boundary and complete the phase slip process.[6]

An analytical treatment of such complicated mechanisms in the real vortex tangle appears to be out of the question. One may note, however, that this problem falls into the currently fashionable category of systems in which a simple but nonlinear set of dynamical rules generates random behavior. Thus it seems natural to adopt an approach which has proved most powerful in this area, namely that of implementing the dynamical ground rules in a numerical simulation.

Before discussing the calculations, we note that Eq. (1) can be put into dimensionless form

$$\partial \vec{s}_0/\partial t_0 = \vec{s}_0' \times \vec{s}_0'' + \alpha \vec{s}_0' \times (\vec{v}_0 - \vec{s}_0' \times \vec{s}_0'') \qquad (1')$$

by measuring lengths in terms of some characteristic distance D, time in units of D^2/β, velocity in units β/D, and force in units $\rho_s \kappa\beta\alpha$. The solution of any problem involving Eq. (1) can be obtained in these reduced units, and then scaled out. This has special implications for the case of homogeneous, steady-state turbulence. Suppose one evaluates the reduced line length density $L_0 = \Omega_0^{-1}\int d\xi_0$ for some driving velocity v_0 by integrating over the line in a sample volume Ω_0 and, if necessary, time averaging. Scaling out to a sampling volume $\Omega = \Omega_0 D^3$, it follows that $L = (\Omega_0 D^3)^{-1}\int D\,d\xi_0 = L_0/D^2$ for $v_{ns} = v_0\beta/D$. However, if the turbulence is homogeneous, L must be independent of the size of the sampling volume, i.e., independent of D. Hence D can be eliminated to yield

$$L = (L_0/v_0^2)v_{ns}^2/\beta^2. \qquad (2)$$

Other average properties have similar scaling relations. For example, the mutual friction force density exerted by the superfluid on the normal fluid,

$$\vec{F}_{sn} = (\rho_s \kappa\alpha/\Omega)\int \vec{s}' \times [\vec{s}' \times (\vec{v}_{ns} - \beta\vec{s}' \times \vec{s}'')]d\xi, \qquad (3)$$

must obey the relation

$$\vec{F}_{sn} = -(\rho_s\kappa\alpha/\beta^2)(F_0/v_0^3)v_{ns}^3\hat{v}_{ns}. \qquad (4)$$

Such functional relations between the steady-state properties and the driving velocity are in fact found experimentally, and are exhibited in a particularly striking fashion in the recent beautiful work of Tough and collaborators.[7]

The problem is now reduced to the evaluation of dimensionless coefficients such as L_0/v_0^2 and F_0/v_0^3. To carry out this evaluation I have developed a computer code which can accurately track the time development of an arbitrarily com-

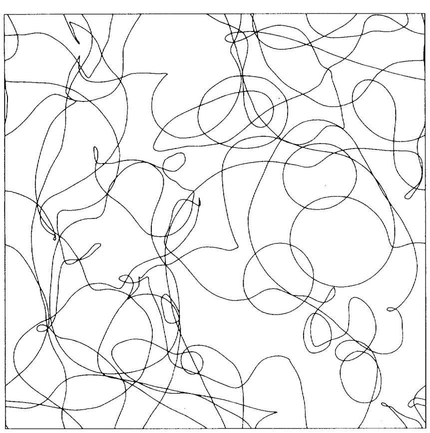

FIG. 2. Vortex tangle generated on the computer, with $\alpha = 0.100$, $v_0 = 40$. The line filling the unit dimensionless cube is projected onto the plane normal to $\hat{v}_0$. The initial configuration from which this configuration evolved consisted of five vortex rings oriented along $\hat{v}_0$.

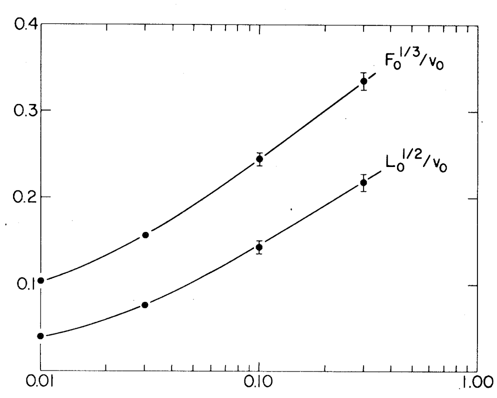

FIG. 3. Computed values of the dimensionless coefficients. The error bars denote the estimated statistical uncertainties in evaluating the steady-state averages. Lines are drawn to guide the eye.

plicated set of vortex singularities using the prescription of Eq. (1′). The code also recognizes line-line crossings, and can make the desired reconnections if instructed to do so. In a typical calculation a sampling volume in the shape of a unit cube with one set of faces perpendicular to $\vec{v}_0$ is filled with an initial nonrandom vortex configuration, which is then allowed to develop in time. Any loop which passes out through one side of the sampling volume reenters it from the opposite side. This is equivalent to starting with an initial condition in which the initial vortex configuration in the sampling volume is periodically repeated throughout all space, and guarantees homogeneity in the large. It is found that the system quickly forgets its initial condition and fluctuates through various states (Fig. 2) in a random manner. The randomness arises from and is maintained by the reconnection events.

Figure 2 can be interpreted as a sample of the vortex tangle, generated entirely by implementing our simple dynamical rules on the computer. L_0, F_0, or any more complicated property of the tangle can readily be evaluated by taking time and volume averages over such configurations as they evolve. Figure 3 shows $L_0^{1/2}/v_0$ and $F_0^{1/3}/v_0$ computed in this way for values of α ranging from 0.01 to 0.30 (corresponding roughly to the temperature range 1.0 to 2.0 K). It is found that the computed ratios are in fact independent of v_0 over a wide range, showing that the calculation indeed produces homogeneous behavior.

The results of Fig. 3 can be compared with the mutual friction measurements of Ref. 7, which seem to represent the most exact experimental realization of the homogeneous turbulent state. In order to achieve an accurate comparison, the β used in scaling out to $\vec{F}_{sn}$ according to Eq. (4) is evaluated by use of an s'' corresponding to the line length density at which these measurements were actually made. Uncertainties in $\alpha(T)$ and in β (e.g., from the core radius) as well as the approximations inherent in Eq. (1) make agreement on the order of 10%–20% the best one could possibly hope for. Figure 4 shows that such agreement with experiment is, in fact, achieved.

I conclude that the approach presented here, with its emphasis on the importance of topology-changing reconnections, provides a satisfactory first-principles description of homogeneous superfluid turbulence. The reconnection concept leads immediately to a plausible model of how the turbulent state sustains itself. The functional dependence of the steady-state properties on the driving velocity arises as a general consequence of the dimensional nature of Eq. (1). Finally, a numerical implementation yields predictions which are in remarkably good agreement with experiment. Unlike previous work, this approach can readily be extended to investigate the effects of boundaries, time-dependent driving velocities, pinning, and so on. Thus it promises to provide a sound basis for further exploration of this interesting class of problems in superfluid hydrodynamics.

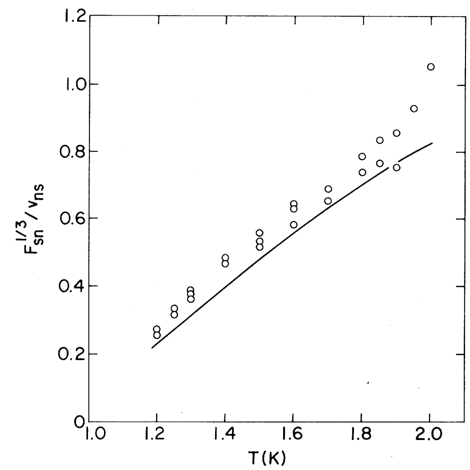

FIG. 4. Comparison of our results with experiment. The curve is obtained by scaling out the upper curve of Fig. 3, using Eq. (4) with $\beta = 1.0\times 10^{-3}$. The points are from Ref. 7.

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