with respect to volume is very large near the separatrix. This does not mean that small-aspect-ratio stellarator fields with good surfaces cannot be found. To find such fields one simply begins with an unimproved field of smaller aspect ratio, which can be obtained by decreasing the helical amplitude $b_{l_0 \cdot m_0}$.

The main tangible result of this work is a method for finding nonaxisymmetric vacuum magnetic fields with significantly increased rotational transform and decreased area of stochasticity and resonances. The problem of calculating the actual coils remains. This calculation is do-able in principle by superimposing coils of various helicities. However, the determination of practical (e.g., modular[9]) coil configurations is a nontrivial problem. The results of Table I are encouraging in this respect. The rapid decrease of the harmonic amplitude with poloidal mode number l indicates that the distortions of present coil designs will not be too rich in harmonic structure.

In addition, this work strongly indicates that there do exist nonaxisymmetric vacuum magnetic fields with a dense set of ergodically covered magnetic surfaces. No proof has been given, but since first-order theory significantly reduces the stochasticity, it is reasonable to believe that infinite-order theory would produce a completely nonstochastic system.

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[1]J. R. Cary and R. G. Littlejohn, to be published.
[2]M. D. Kruskal and R. M. Kulsrud, Phys. Fluids 1, 265 (1958).
[3]H. Grad, Phys. Fluids 10, 137 (1967).
[4]W. A. Newcomb, Phys. Fluids 2, 362 (1959).
[5]K. Miyamoto, *Plasma Physics for Nuclear Fusion* (MIT Press, Cambridge, Mass., 1980), p. 21.
[6]K. Miyamoto, Nucl. Fusion 18, 243 (1978).
[7]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part II.
[8]B. V. Chirikov, Phys. Rep. 52, 263 (1979).
[9]D. T. Anderson, J. A. Derr, and J. L. Shohet, IEEE Trans. Plasma Sci. 9, 212 (1981).

Neutron-Scattering Determination of the Momentum Distribution and the Condensate Fraction in Liquid ^{4}He

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The momentum distribution function, $n(\vec{p})$, is determined for liquid ^{4}He at 1.00, 2.12, 2.27, and 4.27 K from new neutron inelastic-scattering measurements at large momentum transfers. An improved method for extracting the condensate fraction, n_0, from $n(\vec{p})$ is presented and used to obtain new values for n_0 at 1.00 and 2.12 K and a revised value at 1.1 K.

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The problem of demonstrating experimentally that a finite fraction, n_0, of the atoms in superfluid ^{4}He has zero momentum has been of continuing interest ever since London[1] first proposed a connection between the λ transition in liquid ^{4}He and the phenomenon of Bose-Einstein condensation. A recent resurgence of interest[2-5] has centered around the possibility of determining n_0 from the temperature dependence of the pair-correlation function, $g(r)$, via a method proposed by Hyland, Rowlands, and Cummings.[6] Application of this method to $g(r)$ values obtained in a recent neutron-diffraction study[7] has in fact given[2] values of $n_0(T)$ which exhibit a temperature variation of the expected type, and which are consistent with the best theoretical estimates[8-10] of $n_0(0)$. In view of the recent controversy[4] concerning the work of Hyland, Rowlands, and Cummings,[6] it is, however, not clear at present what significance one can ascribe to these values of n_0.

A completely different method, in which n_0 is determined from the momentum distribution function, $n(\vec{p})$, obtained by neutron inelastic-scattering measurements at large wave-vector transfer Q, was proposed even earlier by Hohenberg and Platzman.[11] This method has a rigorous theoretical foundation in the limit $Q \rightarrow \infty$ where the impulse approximation is valid.[12] However, numerous attempts[13,14] to use it at finite Q have led to conflicting estimates of $n_0(T \sim 1.2 \text{ K})$ ranging from 0.02 to 0.17. These disagreements are, we believe, largely attributable to inadequate allowances for the fact that the impulse approximation is not strictly valid in the Q region of the neutron measurements because of distortions caused by final-state interactions and interference effects. The importance of these distortions was first pointed out by Martel *et al.*[14] who also proposed a method for obtaining more reliable values of $n(\vec{p})$ from the neutron measurements. Woods and Sears[13] subsequently applied this method to the neutron results of Cowley and Woods[15] and found that $n_0(1.1 \text{ K}) = 0.069 \pm 0.008$. Application of the improved analysis procedure presented below to the same results gives the revised value 0.109 ± 0.027. This value, the new values for 1.00 and 2.12 K which we present below, the best theoretical values for $n_0(0)$, and the values obtained[2,3] from $g(r)$ are all in very good agreement.

We have recently completed an extensive neutron inelastic-scattering study of liquid ^{4}He at saturated vapor pressure determining, under conditions of very high resolution, the resolution-broadened dynamic structure factor, $S_R(Q,\omega)$, for $Q = 4.0$, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0 Å^{-1} at temperatures of 1.00, 2.12, 2.27, and 4.27 K. Following the procedure of Ref. 13, these $S_R(Q,\omega)$ have been analyzed to obtain the $n(\vec{p})$ shown in Fig. 1. These results represent an average over the five values of Q in the range $5.0 \leq Q \leq 7.0$ Å^{-1} which corresponds very closely to a half period of oscillation of the ^{4}He-^{4}He total atomic scattering cross section,[16] σ, the quantity which governs the final-state interactions.[14] We emphasize that averaging over such a range is necessary to minimize the residual distortions which remain[13,14] even after the $S_R(Q,\omega)$ are symmetrized about the recoil frequencies $\hbar Q^2/2m$.

The horizontal bar in Fig. 1 indicates the half width at half maximum (HWHM) of the instrumental resolution, 0.12 Å^{-1}, which is much smaller than the value of 0.32 Å^{-1} of Ref. 13. The $n(\vec{p})$ distributions of Ref. 13 are thus slightly broader than the corresponding distributions in Fig. 1. Otherwise, there is good agreement between them. Note that there is virtually no change in $n(\vec{p})$ between 4.27 and 2.27 K. However, below $T_\lambda = 2.17$ K there is a large increase at low p which is a direct consequence of a finite condensate fraction. The reason for the surprisingly large increase at 2.12 K (just 0.05 K below T_λ) will be discussed later.

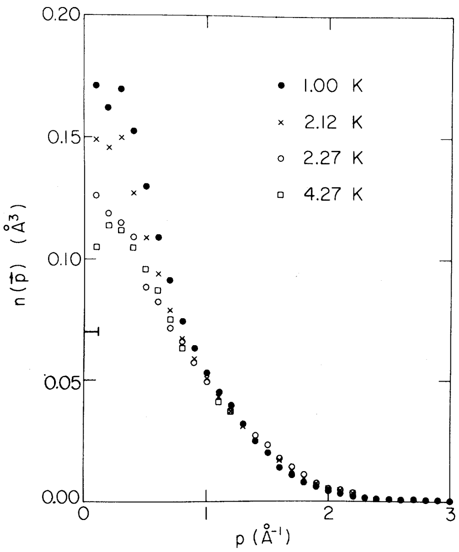

FIG. 1. $n(\vec{p})$ for liquid ^{4}He at four temperatures. The horizontal bar shows the resolution HWHM.

We now introduce an improved procedure for obtaining n_0 from the $n(\vec{p})$. The intrinsic $n(\vec{p})$ is of the form

$$n(\vec{p}) = n_0\delta(\vec{p}) + (1 - n_0)n^*(\vec{p}), \tag{1}$$

where $n^*(\vec{p})$ is the normalized[17] momentum distribution for the uncondensed atoms. The experimental $n(\vec{p})$ are of course broadened both by the instrumental resolution and by the effect of the final-state interactions. The latter contributes a width [full width at half maximum (FWHM)] $\Delta p = \rho\sigma$, where ρ is the number density. For our Q range, Δp is about half the FWHM of $n(\vec{p})$ as a whole and so a distinct condensate peak cannot be resolved. We again note (Fig. 1) that, above $T_\lambda = 2.17$ K where $n_0 = 0$ and $n(\vec{p}) = n^*(\vec{p})$, there is very little dependence on temperature. Since the effect of thermal excitation and thermal expansion on $n^*(\vec{p})$ would be expected to be even smaller below T_λ, it thus seems reasonable to assume that the observed temperature variation of $n(\vec{p})$ below T_λ is mainly due to the change in n_0. We therefore set $n^*(\vec{p})$ equal to the 2.27-K distribution, and it follows from (1) that

$$n_0 = \epsilon/(1 - \beta), \tag{2}$$

TABLE I. The condensate fraction, n_0, and related quantities.

Temperature (K)	p_c (Å^{-1})	ϵ	β	γ	n_0
1.00	1.3 ± 0.1	0.079 ± 0.007	0.49 ± 0.07	0.03 ± 0.01	0.146 ± 0.035
1.1	$1.2_5\pm0.1$[a]	0.069 ± 0.008[a]	0.44 ± 0.07[a]	0.04 ± 0.01	0.109 ± 0.027
2.12	1.1 ± 0.3	0.029 ± 0.011	0.36 ± 0.19	3 ± 1	0.008 ± 0.006

[a] Based on the results of Ref. 13.

where

$$\epsilon = 4\pi \int_0^{p_c} \{n(\vec{p}) - n^*(\vec{p})\} p^2 dp, \tag{3}$$

and

$$\beta = 4\pi \int_0^{p_c} n^*(\vec{p}) p^2 dp. \tag{4}$$

Here p_c is the cutoff point of the broadened condensate peak which we take to be the point where the integrand in (3) goes to zero.

In using the 2.27-K results for $n^*(\vec{p})$, we have neglected the fact that, below T_λ, $n^*(\vec{p})$ is singular as $p\to 0$ with the asymptotic behavior[18]

$$n^*(\vec{p}) = n_0\{ap^{-2} + bp^{-1} + \ldots\}, \tag{5}$$

where $a = mk_BT/8\pi^3\hbar^2\rho n_s$ and $b = mc/16\pi^3\hbar\rho$ (c is the sound velocity and n_s the superfluid fraction, ρ_s/ρ). This singular behavior enhances the apparent value of n_0 as given by (2), and to correct for this enhancement we must replace (2) by

$$n_0 = \epsilon/(1-\beta+\gamma). \tag{6}$$

To evaluate γ precisely, one must know in detail how $n^*(\vec{p})$ is modified below T_λ, and this is not known at present. It is clear, however, that there is a substantial enhancement for temperatures near T_λ, where n_s is small, and a much smaller enhancement for $T\simeq 1.0$ K. This explains why $n(\vec{p})$ for 2.12 K already exhibits an increase at small p which is about half the total increase shown by $n(\vec{p})$ for 1.00 K (see Fig. 1).

To obtain a rough estimate for γ, we assume that the ap^{-2} and bp^{-1} terms in (5) are of importance up to a momentum p' which we take to be the lesser of p_c or the value $p_0 = a/b$ at which the two terms are equal. Integration then gives

$$\gamma = 4\pi(ap' + \tfrac{1}{2}bp'^2), \tag{7}$$

which, for the case $p' = p_0$, becomes

$$\gamma = 3m(k_BT)^2/2\pi^2\hbar^3\rho c n_s^2. \tag{8}$$

The values of p_c, ϵ, β, γ, and n_0 for 1.00, 1.1, and 2.12 K are listed in Table I. In Ref. 13 it was, in effect, assumed that $n_0=\epsilon$, but we see that β is indeed important at all temperatures while γ has an almost negligible effect near 1.0 K, but a very large effect near T_λ.

In Fig. 2 we show $n(\vec{p})$ for 1.00 K (filled circles) and the quantity $(1-n_0)n^*(\vec{p})$ (open circles) where $n_0 = 0.146$. The difference between the filled and open circles represents the broadened condensate peak [see Eq. (1)]. To determine the intrinsic FWHM, Δp, of this peak we must correct for instrumental resolution and the correction depends on the assumed shape of the peak. Averaging over the three temperatures studied, we find that, for a Gaussian shape, $\Delta p = 0.97\pm0.18$ Å^{-1} giving $\sigma = 44\pm8$ Å^2, and, for a Lorentizian shape, $\Delta p = 0.68\pm0.21$ Å^{-1} giving $\sigma = 31\pm10$ Å^2. The good agreement with the known[16] value, 35 ± 2 Å^2, for our Q range strongly supports our procedure for extracting the condensate component from $n(\vec{p})$. The $n(\vec{p})$ for $T=0$ (solid curve in Fig. 2) obtained by Whitlock *et al.*[8] from a Monte Carlo calculation appears to be slightly broader than the experimental distribution.

Figure 3 shows a comparison of our new values (filled symbols) for n_0 with the values (open symbols) obtained[2,3,19,20] from $g(r)$ via the method of Hyland, Rowlands, and Cummings.[6] There is

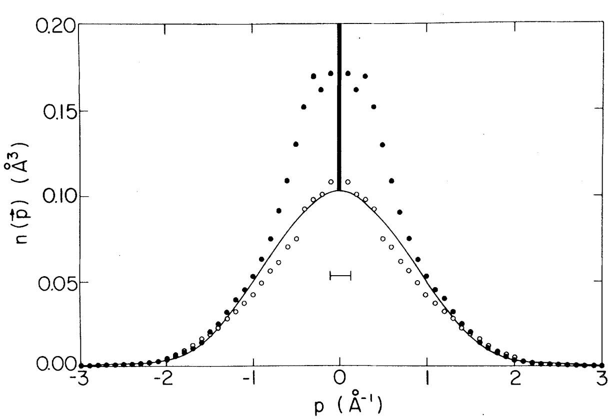

FIG. 2. Comparison of $n(\vec{p})$ (filled circles) and $(1-n_0)n^*(\vec{p})$ (open circles) at 1.00 K. The horizontal bar shows the resolution FWHM. The curve is from a calculation for $T=0$ (Ref. 8).

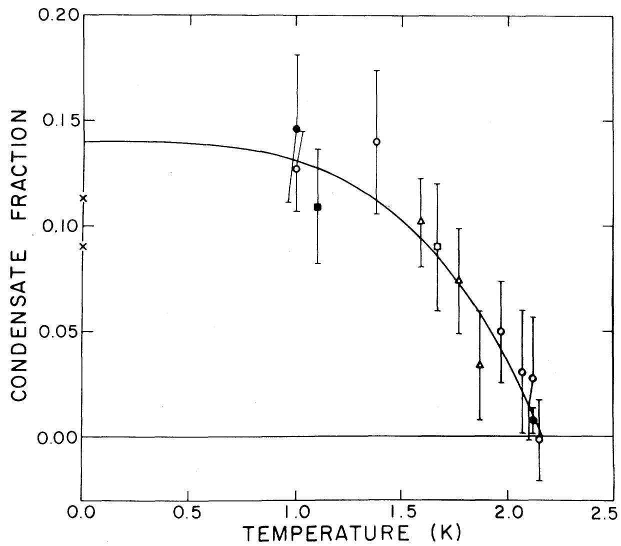

FIG. 3. The condensate fraction in superfluid ^{4}He. Filled symbols show our new results obtained from $n(\vec{p})$. Open symbols are from the temperature variation of $g(r)$: circles (Ref. 2), squares (Ref. 3), and triangles (Ref. 19). The ×'s are from theoretical calculations for $T = 0$: upper (Refs. 8 and 9) and lower (Ref. 10). The curve represents a least-squares fit of Eq. (9) to the experimental values.

clearly very good agreement between the values obtained from the two completely independent approaches which are based on different types of measurements. In view of the existing controversy[4] about the method of Hyland, Rowlands, and Cummings, we cannot, however, rule out the possibility that this agreement is fortuitous.

The solid curve in Fig. 3 is the result of a least-squares fit of the relation

$$n_0(T) = n_0(0)\{1 - (T/T_\lambda)^\alpha\}, \tag{9}$$

to all the experimental values. With $T_\lambda = 2.17$ K, we find that $n_0(0) = 0.139 \pm 0.023$ and $\alpha = 3.6 \pm 1.4$. Most of the many theoretical estimates of $n_0(0)$ lie in the range 0.08 to 0.13. The best values are, we believe, 0.113 (Refs. 8 and 9) and 0.090 (Ref. 10) which are shown by ×'s in Fig. 3. There is no significant difference between these values and our estimate from (9).

In conclusion, we have presented momentum distributions, $n(\vec{p})$, for liquid ^{4}He at 1.00, 2.12, 2.27, and 4.27 K based on new neutron inelastic-scattering measurements at very high experimental resolution. An improved procedure for extracting the condensate fraction, n_0, has been applied to our new $n(\vec{p})$ and to the earlier $n(\vec{p})$ of Ref. 13 to obtain values of n_0 for 1.00, 1.1, and 2.12 K. These values are, we believe, by far the most reliable estimates of n_0 obtained to date. Thus, some 43 years after London's original proposal, there is now strong experimental evidence that a finite fraction (our analysis indicates about 13% at 1 K) of the atoms in superfluid ^{4}He are indeed in the zero-momentum state.

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[1]F. London, Nature 141, 643 (1938), and Phys. Rev. 54, 947 (1938).
[2]V. F. Sears and E. C. Svensson, Phys. Rev. Lett. 43, 2009 (1979), and Int. J. Quantum Chem. 14, 715 (1980).
[3]H. N. Robkoff, D. A. Ewan, and R. B. Hallock, Phys. Rev. Lett. 43, 2006 (1979).
[4]A. Griffin, Phys. Rev. B 22, 5193 (1980); G. V. Chester and L. Reatto, Phys. Rev. B 22, 5199 (1980); A. L. Fetter, Phys. Rev. B 23, 2425 (1981); F. W. Cummings, G. J. Hyland, and G. Rowlands, Phys. Lett. 86A, 370 (1981).
[5]E. C. Sevensson, V. F. Sears, and A. Griffin, Phys. Rev. B 23, 4493 (1981).
[6]G. J. Hyland, G. Rowlands, and F. W. Cummings, Phys. Lett. 31A, 465 (1970); F. W. Cummings, G. J. Hyland, and G. Rowlands, Phys. Koden. Mater. 12, 90 (1970).
[7]E. C. Svensson, V. F. Sears, A. D. B. Woods, and P. Martel, Phys. Rev. B 21, 3638 (1981).
[8]P. A. Whitlock, D. M. Ceperley, G. V. Chester, and M. H. Kalos, Phys. Rev. B 19, 5598 (1979).
[9]P. M. Lam and M. L. Ristig, Phys. Rev. B 20, 1960 (1979).
[10]M. H. Kalos, M. A. Lee, P. A. Whitlock, and G. V. Chester, Phys. Rev. B 24, 115 (1981).
[11]P. C. Hohenberg and P. M. Platzman, Phys. Rev. 152, 198 (1966).
[12]V. F. Sears, Phys. Rev. 185, 200 (1969).
[13]A. D. B. Woods and V. F. Sears, Phys. Rev. Lett. 39, 415 (1977).
[14]P. Martel, E. C. Svensson, A. D. B. Woods, V. F. Sears, and R. A. Cowley, J. Low Temp. Phys. 23, 285 (1976). For extensive coverage of both theoretical and experimental estimates of n_0, see the references cited in this paper and in Refs. 2 and 13.
[15]R. A. Cowley and A. D. B. Woods, Can. J. Phys. 49, 177 (1971).
[16]R. Feltgen, H. Pauly, F. Torello, and H. Vehmeyer, Phys. Rev. Lett. 30, 820 (1973).
[17]We have normalized $n(\vec{p})$ so that $\int n(\vec{p})\,d\vec{p} = 1$ and, hence, $\int n^*(\vec{p})\,d\vec{p} = 1$ as well.
[18]See, e.g., T. H. Cheung and A. Griffin, Can. J. Phys. 48, 2135 (1970); P. C. Martin, Physica (Utrecht) A 96, 70 (1979).
[19]V. F. Sears, E. C. Svensson, and A. F. Murray, to be published.
[20]The value from Ref. 2 for 1.77 K, which is now known to be incorrect (see Footnote 10 in Ref. 5), has been omitted from Fig. 3.