Exact Solution of the Scalar Wave Equation with Focused Gaussian Gain

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An exact analytic solution of the scalar wave equation in a rotationally symmetric focused-Gaussian-amplifying medium is constructed from coupled Gaussian-Laguerre solutions of the free-space wave equation. The solution allows for the analytic construction of the characteristic eigenmodes of the system and has direct application to Raman, dye, and free-electron lasers.

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This Letter addresses an important practical problem of coherent beam propagation in media in which a spatially nonuniform gain coefficient (gain function) is produced by a focused Gaussian beam. This situation arises in laser amplifiers that are optically pumped, such as dye lasers and Raman lasers, and also in free-electron lasers.

A similar problem of propagation in the presence of a transverse quadratic gain variation was previously analyzed by Kogelnik.[1] Cotter, Hanna, and Wyatt[2] applied Kogelnik's quadratic solution to the present case by replacing the Gaussian distribution with a parabolic one. To account for the focused gain function, they used additional approximations. Other authors have also attempted to solve this problem.[3,4] They found the self-growth rates for some of the lower-order free-space modes, but did not consider the full problem including coupled-mode effects which are necessary to properly describe propagation in all but the lowest-gain limit. In this paper we show that if the gain function is proportional to a focused Gaussian distribution, an exact solution of the scalar wave equation can be found.

We begin with the scalar wave equation in the paraxial and slowly-varying-envelope approximation. In the presence of a prescribed gain function at each point in space, $G(x,y,z)$, we have

$$(\nabla_t^2 - 2ik\,\partial_z)\mathcal{E} = -ikG\mathcal{E};\quad \nabla_t^2 = \partial_x^2 + \partial_y^2, \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number of the propagating field and solutions are of the form $E(x,y,z) = \mathcal{E}(x,y,z)e^{i(\omega t - kz)}$. Since G has rotational symmetry, it is natural to use cylindrical coordinates. We expand $\mathcal{E}$ in a complete set of orthonormal Gaussian-Laguerre functions $U_p^l(r,\varphi,z)$ which are referred to as modes of free space[5];

$$\mathcal{E}(r,\varphi,z) = \sum_{p,l} V_p^l(z) U_p^l(r,\varphi,z;k,z_0), \tag{2}$$

where the complex mode amplitudes, $V_p^l(z)$, are functions only of z. The free-space modes, U_p^l, are functions of r,φ,z and depend on the parameter k [set equal to the wave number of Eq. (1)] and on z_0, which is half of what is commonly called the confocal parameter and is left unspecified at this point. In explicit form, U_p^l is given by[5]

$$U_p^l(r,\varphi,z) = \omega^{-1}\left(\frac{p!\,2}{\pi(l+p)!}\right)^{1/2}\left(\frac{\sqrt{2}r}{\omega}\right)^l L_p^l\left(\frac{2r^2}{\omega^2}\right)\exp\left[\left(\frac{-r^2}{\omega^2}\right) - i\left(\pm l\varphi + \frac{kr^2}{2R} - (2p+l+1)\tan^{-1}\frac{z}{z_0}\right)\right], \tag{3}$$

where ω is conventionally called the spot size and is defined by $\omega^2 = (2z_0/k)[1+(z/z_0)^2]$, L_p^l are the associated Laguerre polynomials, and R is the phase-front radius given by $R = z_0[(z/z_0)+(z_0/z)]$. Substitution of Eq. (2) into Eq. (1), multiplication by $U_{p'}^{l'*}(r,\varphi,z)$, and integration over the transverse coordinates, r and φ, gives the following set of ordinary linear coupled differential equations in z for the mode amplitudes. Only modes of the same rotational index l are coupled together by the rotationally symmetric gain.

$$dV_{p'}^l(z)/dz = \sum_p G_{p'p}^l(z) V_p^l(z), \tag{4}$$

where

$$G_{p'p}^l = \tfrac{1}{2}\int_0^{2\pi} d\varphi \int_0^\infty r\,dr\,[U_{p'}^{l*} G(r,z) U_p^l]. \tag{5}$$

The spatial dependence of the gain function is taken as

$$G(r,z) = g_0 |U_0^0(r,z)|^2 = \frac{2g_0\exp(-2r^2/\omega_g^2)}{\pi\omega_g^2}, \tag{6}$$

and the Gaussian spot size of the gain function, ω_g, is given by

$$\omega_g^2 = (2z_0/k_g)[1+(z/z_0)^2].$$

The parameters g_0 and k_g are specified by the

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gain. We now choose the value of z_0 in the field expansion of Eq. (2) as equal to the value established by the prescribed gain function of Eq. (6), and we also choose the two coordinate systems describing the gain function and the propagating field to have coincident origins. Evaluation of the coupling element $G_{p'p}{}^l(z)$ of Eq. (4) then gives

$$G_{p'p}{}^l = (\mu g_0/\pi\omega_g{}^2)\exp[-2i(p'-p)\tan^{-1}(z/z_0)]Q_{p'p}{}^l(\mu), \tag{7}$$

where

$$Q_{p'p}{}^l(\mu) = \mu^l\left(\frac{p'!\,p!}{(p'+l)!\,(p+l)!}\right)^{1/2}\sum_{m=0}^{p}\binom{p+l}{m}\binom{p'+l}{m+p'-p}\frac{(p-m+l)!}{(p-m)!}\mu^{2(p-m)}(1-\mu)^{2m+p'-p}\quad (p'\geq p). \tag{8}$$

The parameter μ, introduced here, is defined by

$$\mu = k/(k+k_g) = \lambda_g/(\lambda_g+\lambda), \tag{9}$$

and measures the overlap of the gain and field intensity distributions. It is limited to the range $0<\mu<1$.

We now make a key substitution of variable for z in Eqs. (4) and (7), namely,[6]

$$\theta = \tan^{-1}(z/z_0),$$

to obtain

$$dV_{p'}{}^l/d\theta = \sum_p M_{p'p}{}^l(\theta)V_p{}^l(\theta), \tag{10}$$

where

$$M_{p'p}{}^l(\theta) = \mu G_p Q_{p'p}{}^l(\mu)\exp[-2i(p'-p)\theta] \tag{11}$$

$(-\pi/2<\theta<\pi/2)$, and where

$$G_p = g_0 k_g/2\pi.$$

We have introduced the quantity G_p to replace g_0 as a descriptive parameter, since G_p corresponds to the plane-wave-field gain coefficient.[7] In explicit matrix form, Eq. (10) is written as

$$dV/d\theta = \underline{M}(\theta)V(\theta). \tag{12}$$

To simplify the notation, we have dropped the common index l. From the θ dependence of Eq. (11), we can see that the matrix $\underline{M}$ is of the form

$$\underline{M}(\theta) = \underline{U}^\dagger(\theta)\underline{K}\underline{U}(\theta),$$

where $\underline{U}$ is a unitary matrix of the form $\underline{U}(\theta) = \exp(i\underline{H}\theta)$, with $\underline{H}$ a constant diagonal (Hermitian) matrix whose elements are given by $H_{mn} = 2m\delta_{mn}$.[8] Moreover, $\underline{K}$ is a constant, real symmetric matrix with elements

$$K_{p'p}{}^l = G_p Q_{p'p}{}^l(\mu).$$

For convenience, we introduce the vector $y(\theta) = \underline{U}(\theta)V(\theta)$. After some algebraic manipulation, Eq. (12) becomes

$$dy/d\theta = (\underline{K}+i\underline{H})y(\theta), \tag{13}$$

$\underline{K},\underline{H}$ independent of θ. The formal solution for y is

$$y(\theta) = [\exp(\underline{K}+i\underline{H})(\theta-\theta_0)]y(\theta_0), \tag{14}$$

and the solution for V is

$$V(\theta) = \underline{U}^\dagger(\theta)\exp[(\underline{K}+i\underline{H})(\theta-\theta_0)]\underline{U}(\theta_0)V(\theta_0). \tag{15}$$

For all cases we have considered, an explicit solution for $V(\theta)$ is available through diagonalization, that is,

$$\underline{S}^{-1}(\underline{K}+i\underline{H})\underline{S} = \underline{D}, \tag{16}$$

where $\underline{D}$ is a constant diagonal matrix of $\underline{K}+i\underline{H}$, and $\underline{S}$ is a constant nonsingular matrix whose columns are the corresponding eigenvectors.

The eigenmodes of the electromagnetic system with gain are found by choosing the vector V to be such that the corresponding vector y is an eigenvector of the matrix $\underline{K}+i\underline{H}$. The amplitude of an eigenmode field $\mathcal{E}(r,\theta)$ reproduces itself at different values of the propagation variable θ in precisely the same sense as do the free-space modes. That is, it can be shown that the eigenmode field

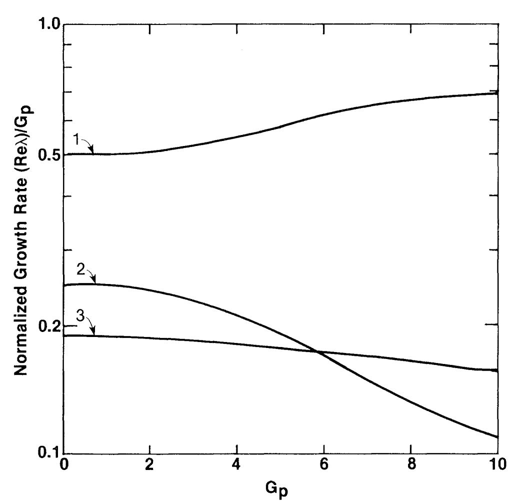

FIG. 1. Normalized real parts of eigenvalues vs plane-wave gain coefficient G_p, for $\mu = 0.5$.

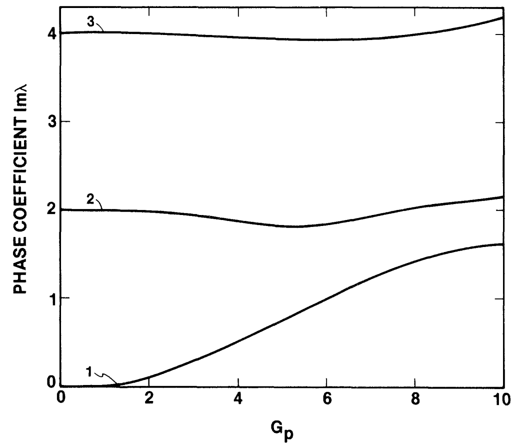

FIG. 2. Imaginary parts of eigenvalues vs plane-wave gain coefficient, G_p, for μ = 0.5.

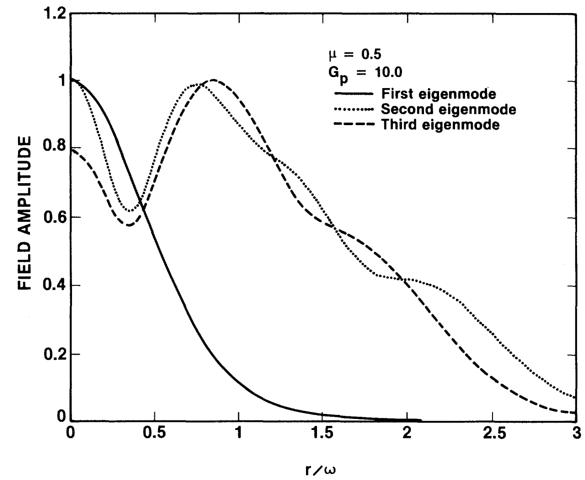

FIG. 3. Magnitudes of eigenmode fields vs normalized radial coordinate.

amplitudes at two different values of θ are identical complex functions of r/ω when the transverse coordinate is measured along surfaces whose curvature is equal to that of the free-space modes. The two fields then differ only by a normalization factor and the propagation factor $\exp[(\lambda+i)(\theta_1-\theta_2)]$, where λ is the corresponding complex eigenvalue. When the gain vanishes the eigenvalue λ equals $2pi$, and the free-space result is recovered.

We have numerically evaluated the eigenvalues and eigenvectors for the case $\mu=0.5$, as a function of G_p, by truncation of the matrix $\underline{K}+i\underline{H}$ to forty rotationally symmetric ($l=0$) free-space modes. It can be shown that the rotationally symmetric modes dominate all higher-order rotational modes in growth rate. The value of the parameter $\mu=0.5$ corresponds to its upper limit for Raman-Stokes emission and is approximately representative of most uv and visible Raman lasers.

Figures 1 and 2 are plots of the real and imaginary parts of the first three eigenvalues as a function of the plane-wave gain G_p. The real parts can be identified as the growth rates of the eigenmodes and have been plotted normalized to the plane-wave gain. As can be seen in Fig. 1, the first-eigenmode growth dominates that of the others by at least a factor of 2 in the exponential of Eq. (15). In fact, for values of $G_p \sim 10$, corresponding to superfluorescent single-pass Raman lasers, the dominance is sufficient to make the output field independent of the input distribution and proportional only to the first-eigenmode field. Figures 3 and 4 are plots of the eigenmode field magnitudes and phase distributions as functions of the normalized radial coordinate, r/ω, for $G_p=10$. The phase plot of Fig. 4 represents the deviation from the free-space mode curvature. It should be noted that the spatial profile of the first-eigenmode field magnitude is approximately a Gaussian distribution with a waist smaller than the $U_0{}^0$ mode; this narrowing is qualitatively related to the phenomenon of "gain focusing" that occurs for parabolic gain distributions.[1,2]

In conclusion, we have presented an exact solution for wave propagation in the presence of focused Gaussian gain. The solution is of practical utility for the rapid evaluation of a large class of propagation problems that includes Raman,

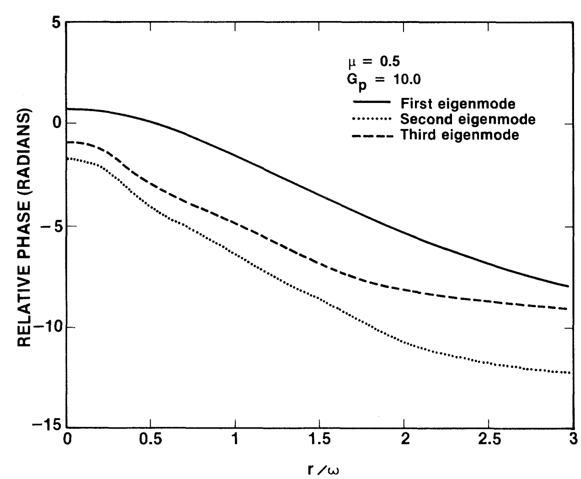

FIG. 4. Eigenmode phase deviations from free-space mode curvature vs normalized radial coordinate.

dye, and free-electron lasers. The same method can be used to treat cases of focused-Gaussian-refractive-index media and of partially filled resonators. It can also be used when the gain or index distribution is proportional to the magnitude squared of any single Gaussian-Laguerre mode function. In future publications we will present results for these cases and extend the treatment to multiple-pass amplifiers.[9]

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[7]The plane-wave-field gain coefficient is usually defined for Raman lasers as the exponential growth rate that would apply if the peak gain coefficient were uniform in the transverse dimension as would occur in the limit $k_g \rightarrow 0$ or $\mu \rightarrow 1$.

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Statistics of Millimeter-Wave Photons Emitted by a Rydberg-Atom Maser: An Experimental Study of Fluctuations in Single-Mode Superradiance

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An experimental study of the fluctuations of a millimeter-wave transient Rydberg-atom maser is presented. The photon-number probability distribution is shown to evolve in time from a Bose-Einstein to a bell-shaped distribution. The experiment is a quantitative check of single-mode superradiance theory.

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Rydberg atoms, excited by a laser pulse inside a properly tuned millimeter-wave cavity, amplify the radiation noise at the cavity resonant frequency and emit a transient maser pulse.[1] By use of a very sensitive detection technique based on atomic field ionization, it is possible to actually count the atoms which have radiated during a given time interval inside the cavity, which obviously amounts to counting the number of photons emitted during that time. Such a photon-counting type of experiment is quite novel in this part of the radiation spectrum. It allows us to test under almost ideal conditions the simplest model of superradiance (the so-called "single-mode" model[2]). In this Letter, we present an experimental study of the pulse-to-pulse fluctuations of a Rydberg maser made of a few thousand radiators. We have observed how the histogram of the number n of emitted photons qualitatively changes during the emission time: Typical of a Bose-Einstein field at the beginning of the process, it evolves at later time into a broad bell-shaped curve presenting a maximum peak for a value $n \neq 0$. The measured histograms have been found to be in very good agreement with the predictions of the "single-mode" superradiance theory. This is to our knowledge the first direct and quantitative test of the theory, in a system in which superradiance is not complicated—as is the case in the optical domain—by propagation or diffraction effects.[3]

The experimental setup is sketched in Fig. 1(a): A thermal beam of Na atoms is excited by a dye-laser pulse (5 ns duration) to the $29S_{1/2}$ level inside a millimeter-wave semiconfocal Fabry-Perot cavity tuned to resonance with the transition towards the less excited $28P_{1/2}$ level ($\nu = 162.4$ GHz). The cavity has an intermirror length $L = 1.2$ cm and a $Q = 10\,000$. It sustains a mode with a Gaussian transverse profile (waist $w_0 = 2.8$ mm). The small atomic sample is located at an antinode position in the mode standing-wave pattern, so that all atoms are *equivalently* coupled to the field. The atoms interaction with the cavi-

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