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Stochastic Instability of Sine-Gordon Solitons

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A breather soliton of the sine-Gordon equation is shown to behave stochastically in the presence of an external oscillating field, which leads to random pair creations of kink and antikink solitons.

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Recently there has been a surge of interest in statistics of solitons in various physical fields.[1,2] Since nonlinear wave equations governing solitons are completely integrable, there must be some mechanisms which enable us to consider a statistical ensemble of solitons. For nontopological solitons the thermal mechanism may be natural since its activation energy can be very small. But for phase solitons such as kinks of the sine-Gordon (SG) equation, which have a finite creation energy, nonthermal excitation mechanisms will play an important role, especially in nonequilibrium systems. In this Letter, I show that kink solitons of the SG equation can be randomly created through the stochastic instability of breathers in the presence of an oscillating external field. This system is the model of many physically interesting phenomena such as charge-density waves in a one-dimensional condensate in the presence of an ac electric field,[3] the magnetic-flux propagation on a Josephson-junction transmission line with an ac external current,[4] and the creation of baryons from a meson through the interaction with an external oscillating field.[5]

The interaction between SG solitons and an external field is described by

$$u_{tt} - u_{xx} + \sin u = \epsilon \cos\omega t \equiv f(t), \qquad (1)$$

where ϵ is a small parameter and $f(t)$ is an external field. The Hamiltonian of Eq. (1) is

$$H = \tfrac{1}{2}\int dx[u_t^{\,2} + u_x^{\,2} + 2(1-\cos u)] - f\int u\,dx, \qquad (2)$$

while the total momentum of the field $P \equiv -\int u_x u_t \times dx$ is still conserved when $u(\infty,t) - u(-\infty,t) = 0$ which is satisfied for a breather and a kink-antikink pair. Therefore we can choose a reference frame such that the center of mass of a breather (or a kink-antikink pair) is at rest even in the presence of the external field. This is equivalent to eliminating one degree of freedom of a breather and enables us to perform a simple perturbational approach.

If $\epsilon = 0$, Eq. (1) is completely integrable in terms of canonical variables which consist of the scattering data of the associated linear eigenvalue equation,[6] and it has the following soliton solution:

$$u = 4\arctan[T(t)\,\mathrm{sech}\,2k(x - x_0)], \qquad (3)$$

where x_0 is constant and $T(t) = \sin(\alpha/16)\tan\theta$, $k = \frac{1}{2}\sin\theta$ for a breather and $T(t) = (e^p + e^{-p})\sinh(q/16)/(e^p - e^{-p})$, $k = e^p + e^{-p}$ for a kink-antikink pair. Here (θ, α) and (p, q) are the canonical conjugate variables which are given by

$$\theta(t) = \theta(0),\ \ p(t) = p(0),$$

$$\alpha(t) = 16t\cos\theta + \alpha(0), \quad q = 32(e^p - e^{-p})t + q(0),$$

where the unperturbed Hamiltonian is given by $H_0 = 16\sin\theta$ or $H_0 = 32(e^p + e^{-p})$. Since the long-time behavior of θ, p is of interest in the presence of the external oscillating field, the averaging method for Hamiltonian systems can be applied. Substituting Eq. (3) into Eq. (2), we have the approximate Hamiltonian

$$H = H_0 + (2\pi f/k)\ln[-T + (1 + T^2)^{1/2}], \qquad (4)$$

and the canonical equations yield

$$\mu_t{}^2 = \frac{\pi}{2}\,\frac{f(1-\mu^2)^{3/2}(1+\mu^2\nu^2/4)^{1/2}}{(1+\nu^2/4)^{1/2}}, \tag{5}$$

$$\nu_t = \frac{\mp 2(1+\mu^2\nu^2/4)^{1/2}}{(1-\mu^2)^{1/2}}\left[1-\frac{\pi f}{4(1-\mu^2)}\ln\left\{\frac{\nu}{2}+\left(1+\frac{\nu^2}{4}\right)^{1/2}\right\}\right], \tag{6}$$

where $\nu = -2T$, $\mu^2 = -\cot^2\theta < 0$ for a breather, and $\mu^2 = (e^p - e^{-p})^2/(e^p+e^{-p})^2 > 0$ for a kink-antikink pair. Equations (5) and (6) can also be derived by the perturbation method of Kaup and Newell.[7]

In order to express the oscillating character of a breather explicitly, we transform Eq. (6) to

$$\psi_t = -\cos\theta + \frac{H_1}{16\cot\theta} - \frac{\pi f \sin\psi}{64\sin\theta\cos^2\theta(1+T^2)^{1/2}} \tag{7}$$

where $\psi = -\alpha/16$ and H_1 is the perturbed Hamiltonian given in Eq. (4). Thus, a breather is an oscillating bound state of a kink-antikink pair, whose frequencies consist of a fundamental energy-dependent frequency $-\cos\theta$ and its odd higher harmonics $-(2n+1)\cos\theta$. The external oscillating field interacts with a breather resonantly at each harmonic and the overlapping of these resonances leads to the chaotic behavior of a breather.

Let us derive the approximate criterion for the overlapping of resonances[8] by means of the Hamiltonian (4), which turns out to be

$$H = 16\sin\theta + \frac{4\pi\epsilon}{\sin\theta\cos(\omega t)}\sum_{n=0}^{\infty} b_{2n+1}\sin(2n+1)\psi,$$

where $\psi \approx -t\cos\theta$ and $\{b_{2n+1}\}$ are the Fourier-sine coefficients of $\ln[\tan\theta\sin\psi + (1+\tan^2\theta\sin^2\psi)^{1/2}]$. If $|\tan\theta| \ll 1$, that is, the energy of a breather $(16\sin\theta)$ is small, the Hamiltonian is reduced to

$$H \approx 16\theta + 4\pi\epsilon\cos(\omega t)\sin\psi,$$

which gives a single resonance at $\omega \approx \cos\theta \lesssim 1$. This case was discussed by Kaup and Newell in connection with a charge-density wave in a one-dimensional condensate.[9] For a high-energy breather such that $|\tan\theta\sin\psi| \gg 1$ is satisfied, we have

$$H \approx 16\sin\theta - \frac{4\pi\epsilon\cos(\omega t)\,|\sin\psi|}{\sin\psi\sin\theta}\left[\ln(\tan\theta) + \sum_{m=1}^{\infty} m^{-1}\cos(2m\psi)\right],$$

where $\ln(\tan\theta)$ and m^{-1} correspond to $-b_1$ and $-b_{2m+1}$, respectively. When ϵ is small enough, the trajectories in the phase space (θ, ψ) consist of many separated resonance islands, corresponding to each resonance $\omega = (2n+1)\cos\theta$. Since $|\cos\theta| \ll 1$ in this case, the resonances occur at large n (or m). The width of each resonance island can be estimated as $\delta\theta_m \approx (\pi\epsilon/m)^{1/2}$ whereas the distance between neighboring islands is given by $\Delta\theta_m \approx \omega/2m^2$. The overlapping condition of neighboring islands, that is, $\delta\theta_m/\Delta\theta_m > 1$, yields

$$m > (\omega/2)^{2/3}(\pi\epsilon)^{-1/3},$$

which gives the stochastic region in the phase space:

$$\pi/2 > \theta > \pi/2 - (\pi\epsilon\omega/2)^{1/3}, \tag{8}$$

where $\theta = \pi/2$ is the boundary between a breather and a kink-antikink pair. From Eq. (8) it is found that the amount of energy gain through this stochastic process is roughly of the order $\epsilon^{1/3}$ which is larger than that due to a single resonance $(\epsilon^{1/2})$. It is also expected that a breather in the stochastic region gains energy stochastically and splits into a kink-antikink pair randomly. This is confirmed by numerical integrations of Eqs. (5) and (6) and a typical numerical result is shown in Fig. 1. In this example, 31 points out of 54 fall into the kink-antikink region $(\mu^2 > 0)$ from the breather region $(\mu^2 < 0)$ through "holes" $(\mu^2 = 0,\ \psi = 0, \pi, 2\pi)$ in the phase space. These holes are on the path through which a breather solution is analytically continued to a kink-antikink solution and Fig. 1 shows relatively regular trajectories near the holes. However, a kink-antikink pair is chaotically created in the sense that the variable $T(t)$ or $\nu(t)$ governing actual behaviors of a kink-antikink pair is not determined regularly at the holes [see Eq. (7) and the definition of T]. After a kink and an antikink are created by such a random pair production, they behave coherently like relativistic particles changing their momentum and energy periodically as a result of the external field. In other words, kinks are created in the rarefied gas of randomized breathers as coherent structures. This mechanism of chaotic creation of a kink-

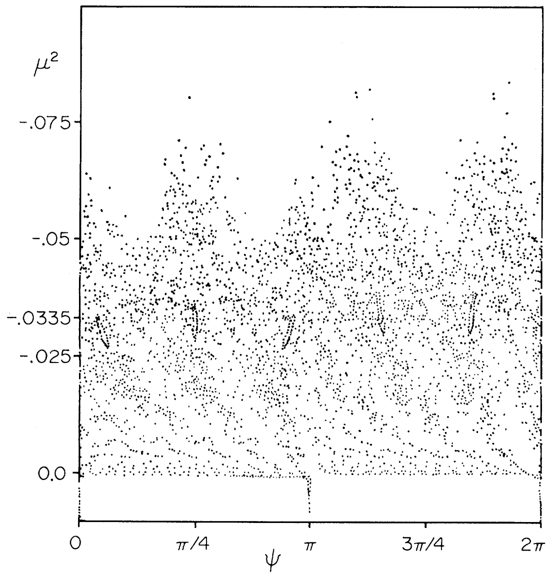

FIG. 1. Trajectories of phase points resulting from 54 initial values on the line $\mu^2 = -0.0335$, which is close to the resonance, $n = 3$, for $\epsilon = 0.01$ and $\omega = 0.9$. Points are plotted at time intervals $2\pi/\omega$ until $t = 240\times 2\pi/\omega$. The width of the stochastic region is estimated from Eq. (8), which gives $0 > \mu^2 > -0.06$. This estimation reasonably explains the numerical result except in the narrow core region of the resonance, $n = 3$.

antikink pair provides the theoretical interpretation of the recent numerical investigation.[10]

In the above discussion, the effects of excitation of a nonsoliton field are neglected, which is justified for the case $\omega < 1$. In this case the nonlocal excitation of nonsoliton fields by the direct resonance does not occur because their frequencies are greater than 1. Although small nonsoliton fields are excited locally near a breather through resonances between oscillations of nonsoliton fields and the coupled oscillation of a breather and the external field, their contribution to the perturbed Hamiltonian $f\int u_{ns}\,dx$ is smaller than $f\int u_s\,dx$, where u_{ns} and u_s are the nonsoliton and solition fields, respectively. Finally, it is noted that a low-energy breather also becomes stochastic if the external field consists of many different frequency components instead of a single one, which will be discussed elsewhere.

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Necessary Conditions on Potential Functions for Nonrelativistic Bound States

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It is shown that $\int|V(\vec{x})|^{3/2}d^3x > (\pi^2/4)(3\hbar^2/2m)^{3/2}$ is a necessary condition for one or more normalizable bound-state solutions to the nonrelativistic Schrödinger energy-eigenvalue equation for any potential function $V(\vec{x})$ such that the integral is finite. Moreover, for potentials such that $\int|V(\vec{x})|^{3\gamma/2}d^3x$ is finite for a value or values of $\gamma > 1$, the magnitude of a negative energy eigenvalue is related to the latter integral by a general inequality derived here.

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Veling[1] has recently reported an important result for the nonrelativistic one-dimensional Schrödinger energy-eigenvalue equation with a potential function such that $\int_{-\infty}^{\infty}|V(x)|^{\gamma}dx$ is finite for some constant parameter value(s) $\gamma \geq 1$: Negative energy eigenvalues associated with normalizable bound states are

