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Stochasticity Threshold for Hamiltonians with Zero or One Primary Resonance

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The large-scale stochasticity threshold for Hamiltonians with two degrees of freedom with only one primary resonance can be analytically estimated because of the quite steep growth of the stochastic layer of this resonance. For Hamiltonians without primary resonance, the threshold is computed by available techniques after canonical transformations. Thus the first analytical estimate of the threshold is obtained for the Hénon-Heiles Hamiltonian and an example of Walker and Ford.

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The aim of this Letter is to fill a gap in the available methods for computing large-scale stochasticity (LSS) thresholds in Hamiltonian systems with two degrees of freedom. These thresholds are all-important quantities in various problems (see for instance Chirikov[1] and references therein). In many cases the Hamiltonian can be written as

$$H(\vec{I},\vec{\theta})=H_0(\vec{I})+\epsilon\sum_{\vec{n}\in\mathscr{I}} V_n(\vec{I})\cos(\vec{n}\cdot\vec{\theta}+\chi_n), \qquad (1)$$

where $\mathscr{I}$ is some set of couples of integers, χ_n is a phase, and $\vec{I}=(I_1,I_2)$ and $\vec{\theta}=(\theta_1,\theta_2)$ are the action-angle variables of H_0. Generically, for $\epsilon\neq 0$ H is no longer integrable. Then regions of chaotic motion[1,2] appear in phase space and their size increases with ϵ. Merging of such regions can lead to the appearance of LSS.[1,2] Time-dependent Hamiltonians with one degree of freedom can easily be written[3] in the form of Eq. (1).

The term with index $\vec{n}$ of Eq. (1) is said to be resonant[1,2] for a given value of the energy E if there is an action value $\vec{I}_n=(I_{n_1},I_{n_2})$, such that $E=H_0(\vec{I}_n)$ and

$$\vec{n}\cdot\vec{\omega}(\vec{I}_n)=0, \qquad (2)$$

with $\vec{\omega}=\partial H_0/\partial\vec{I}$. This means that $\vec{n}$ is tangent to the unperturbed energy curve at $\vec{I}_n$ [see Fig. 1(a)] and that, for small values of ϵ, the phase $\vec{n}\cdot\vec{\theta}$ is stationary since $d\vec{\theta}/dt=\partial H/\partial\vec{I}$. Then H is said to have the primary resonance $\vec{n}$.

Here we consider the case where H has fewer than two primary resonances. When H has no resonance we show that canonical transformations always allow us to get a new Hamiltonian H', with at least one primary resonance. For the case when H' has at least two primary resonances, there already exist methods to compute the threshold of LSS.[1-4] When H or H' has only one primary resonance, LSS corresponds[4] to the sudden blowup of the stochastic layer[1] of this resonance. The analytical description of this blowup[1,5] gives a method for computing the threshold of LSS. We illustrate the method by applying it to the Hamiltonian

$$H_w(\vec{I},\vec{\theta})=H_0(\vec{I})+0.95 I_1 I_2\cos(2\theta_1-2\theta_2)+0.25 I_1^{3/2} I_2\cos(3\theta_1-2\theta_2), \tag{3}$$

where

$$H_0(\vec{I})=I_1+I_2-I_1^2-3I_1I_2+I_2^2,$$

introduced by Walker and Ford.[4] In this case, $\epsilon=1$ but the amplitude of cosine terms is governed by the total energy E. The $(2,-2)$ resonance always exists but the $(3,-2)$ one appears for $E_w \simeq 0.08$. At that value, the Poincaré map of the system exhibits an appreciable chaotic domain in the neighborhood of the $(2,-2)$ resonance. All the various methods based on the existence of the two primary resonances would predict a threshold larger than E_w. Our method, which applies to the case of only one resonance, here labeled by $(2,-2)$, yields a threshold $E_w' \simeq 0.078$ consistent with the numerical results. We also consider the Hamiltonians

$$h^{\pm}(x_1,x_2,y_1,y_2) = (x_1^2+x_2^2+y_1^2+y_2^2)/2+x_1^2x_2\pm x_2^3/3. \tag{4}$$

h^- is the Hénon-Heiles Hamiltonian[6] and h^+ is an integrable Hamiltonian.[7] Figure 7 of Ref. 6 shows the rapid increase of the area of the chaotic region in Poincaré maps of h^- when the energy E increases and it yields a numerical threshold $E_n^-=0.11$. We compute here the first theoretical estimates of this threshold by two different ways which use the methods of this Letter and, respectively, yield $E_t^-=0.08$ (which agrees within 27% with E_n^-) and $E_t^{-\prime}=0.106$ (which agrees within 4% with E_n^-). Hamiltonian h^+ has an energy of escape $E_e^+=\frac{1}{12}$ above which the equipotential lines open and phase space is no longer bounded. Forgetting that h^+ is integrable, we perform the same two calculations for h^+. We get $E_t^+=0.115$ and $E_t^{+\prime}=0.15$ which are both above E_e^+. Since LSS is defined[1-4] for bounded phase spaces, this result is a hint of the integrability of h^+. We now describe our general methodology before applying it to H_w and $h^\pm$.

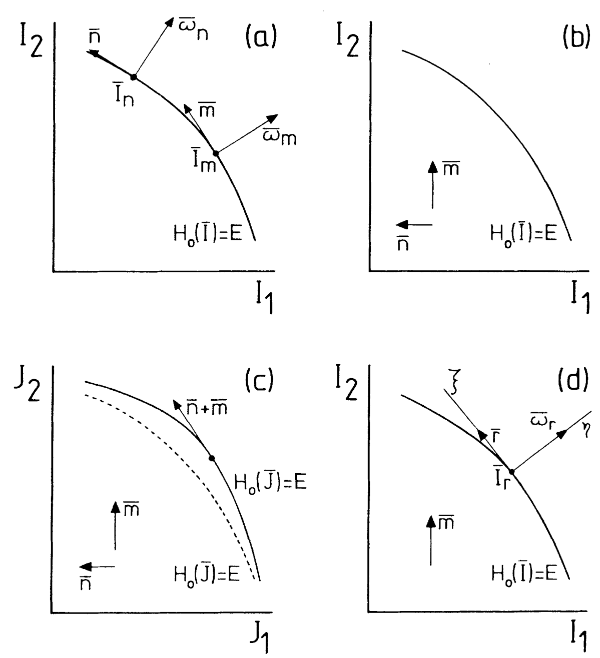

FIG. 1. Resonant directions and energy curve. (a) Two primary resonances. (b) No primary resonance. (c) Case (b) after one canonical transformation; the old (new) energy curve is the dashed (continuous) curve; the resonant $\vec{m}+\vec{n}$ direction is now created. (d) One unique primary resonance; new axes ξ and η are defined.

We assume first that H has *no primary resonance* [the case of Fig. 1(b), in which for clarity $\mathcal{S}$ contains only two elements labeled $\vec{m}$ and $\vec{n}$]. In order to generate terms corresponding to secondary resonances, we perform on H a Kolmogorov transformation[1,2] which kills all the cosine terms of Eq. (1) to order ϵ. This transformation goes from the $(\vec{I},\vec{\theta})$ to the $(\vec{J},\vec{\varphi})$ coordinates and is defined by the generating function

$$F(\vec{J},\vec{\theta})=\vec{J}\cdot\vec{\theta}-\epsilon\sum_{\vec{n}\in\mathcal{S}}\frac{V_n(\vec{J})\sin(\vec{n}\cdot\vec{\theta}+\chi_n)}{\vec{n}\cdot\vec{\omega}(\vec{J})}.$$

The absence of primary resonance precludes any problem of a small denominator. The new Hamiltonian H' contains terms of order ϵ^i, $i\geq 2$. The terms of order ϵ^2 may be written in the form

$$\epsilon^2 W_{mn}(\vec{J})\cos[(\vec{m}\pm\vec{n})\cdot\vec{\varphi}+\chi_m\pm\chi_n],$$

where $\vec{m}$ and $\vec{n}$ belong to $\mathcal{S}$. Those with $\vec{m}=\vec{n}$ and a minus sign enter into the angle-independent part H_0' of H' and modify the range of directions of $\vec{\omega}'=\partial H_0'/\partial\vec{J}$ with respect to that of $\vec{\omega}$. The directions of the set of vectors $\vec{m}\pm\vec{n}$ cover a wider range than those of $\mathcal{S}$. This fact and possibly the increased range of directions of $\vec{\omega}'$ make more likely the existence of at least one primary resonance for H' [see Fig. 1(c)]. Moreover, the $O(\epsilon^3)$ terms also contribute to modify H_0 and to create new possible resonant directions. Higher-order terms cannot be retained after this first canonical transformation, since the actual amplitude of the corresponding resonances is modified by the $O(\epsilon^2)$ and $O(\epsilon^3)$ terms. Therefore, if there is no $O(\epsilon^2)$ or $O(\epsilon^3)$ primary resonance, the process must be iterated by killing these terms by a new Kolmogorov transformation. If iterations are necessary for getting at least one resonant term, this term may be of order between ϵ^{2^l} and $\epsilon^{2^{l+1}-1}$ and higher-order terms must not be considered for the same reason as previously. If only one primary resonance appears after these iterations, the method of the next paragraph must be used. If more resonances are present, classical methods[1-4] may be applied. A concrete example where H has no resonant term is given later with the first treatment of h^*.

We now assume that H has a *unique resonant term* $\vec{n}=\vec{r}$. Then LSS appears as a result of the blowup of the stochastic layer of resonance $\vec{r}$. Our method consists in approximately reducing H in the vicinity of resonance $\vec{r}$ to the time-dependent Hamiltonian

$$H_p(v,x,\tau)=v^2/2-M\cos x-P\cos k(x-\tau).$$

The width w of the stochastic layer of the resonance with amplitude M has been computed in Ref. 5 which generalized the results of Ref. 1 obtained for $M/P=k=1$. It yields $w\propto P\exp(-1/\rho)/M\rho^{2k+1}$ where $\rho=2M^{1/2}/\pi k$, $\rho\ll 1$. If P/M is of order ρ^s, then

$$w\propto\rho^{-\lambda}\exp(-1/\rho), \tag{5}$$

where $\lambda=2k+1-s$. At $\rho_i=[1-1/(\lambda+1)^{1/2}]/\lambda$, $w(\rho)$ has an inflection point. The very steep rise of $w(\rho)$ leads us to define a threshold of blowup of the stochastic layer as the value $\rho_s=[\lambda+2-2(\lambda+1)^{1/2}]/\lambda^2$ of ρ where the tangent at ρ_i cuts the ρ axis. Two approximations allow us to reduce H to H_p. First we make a Taylor expansion of H in $\vec{I}$ in the vicinity of $\vec{I}_r$ and replace the $V_n(\vec{I})$'s by the constant $V_n(\vec{I}_r)$'s and keep the terms of the expansion of $H_0(\vec{I})$ to second order.[3] We then go to new action variables (ξ,η), such that their axes are aligned with the tangent and the normal to the energy curve at $\vec{I}_r$ [see Fig. 1(d)]. These variables are defined by the generating function $F'(\xi,\eta,\vec{\theta})=\vec{\theta}\cdot(\vec{I}_r+\xi\vec{r}+\eta\vec{\omega}_r)$, where $\vec{\omega}_r=\vec{\omega}(\vec{I}_r)$. The new angle variables are $x=\partial F'/\partial\xi=\vec{r}\cdot\vec{\theta}$ and $y=\partial F'/\partial\eta=\vec{\omega}_r\cdot\vec{\theta}$. Let (α_n,β_n), $\vec{n}\in\mathcal{G}$, be defined by $\vec{n}=\alpha_n\vec{r}-\beta_n\vec{\omega}_r$. Then $\vec{n}\cdot\vec{\theta}=\alpha_n x-\beta_n y$. For not too high values of ϵ, motion occurs in the vicinity of the unperturbed energy line. Thus η is of order ξ^2 and the $\xi\eta$ and η^2 terms are neglected. This yields a new Hamiltonian

$$H'(\xi,\eta,x,y)=a\xi^2/2+\omega_r^2\eta+\sum_{\vec{n}\in\mathcal{G}}U_n\cos(\alpha_n x-\beta_n y+\chi_n),$$

where $a=\vec{r}\cdot\vec{\sigma}_r\cdot\vec{r}$, with $\vec{\sigma}_r=\partial\vec{\omega}/\partial\vec{I}(\vec{I}_r)$, $U_n=\epsilon\times V_n(\vec{I}_r)$; the constant term $H_0(\vec{I}_r)$ has been deleted since it plays no role in the dynamics. We now proceed to the second approximation and retain in the sum of H' only two terms, the one previously labeled r that describes the primary resonance of H and one of the other elements of $\mathcal{G}$. Let this last term be labeled by m. The question of the right choice of m will be solved later. We then obtain an approximate Hamiltonian that is written

$$H''(\xi,\eta,x,y)=a\xi^2/2+\omega_r^2\eta+U_r\cos x+U_m\cos(\alpha_m x-\beta_m y),$$

where the constant phases χ_n and χ_m have been dropped, since they can be suppressed by a simple change of the origin of x and y. The canonical equations of H'' allow us to write a second-order differential equation for x which is the same as the one obtained from the canonical equations for H_p with $\tau=\beta_m\omega_r^2t/\alpha_m$, $k=\alpha_m$, $M=b|U_r|$, $P=b|U_m|$ where $b=|a|\alpha_m^2/(\beta_m^2\omega_r^4)$. According to Eq. (5) the largest blowup of the stochastic layer of resonance M is given by the m term in H' which yields λ maximum and consequently corresponds to α_m maximum. The minimum value of ϵ such that $\rho=2|U_r a|^{1/2}/(\pi|\beta_m|\omega_r^2)$ equals ρ_s gives the threshold for LSS. As $|U_r a|$ is a growing function of ϵ, the threshold corresponds to the minimum value of β_m. Generally this leads to the retention in the sum of H' the m term for which α_m/β_m is maximum and consequently such that $\vec{m}$ has the direction closest to $\vec{r}$. This reduction of H to H_p makes sense only if the nonlinearity of H is mostly borne by H_0, that is, if the perturbed motion remains close to the unperturbed energy line. A necessary condition for this is $a\gg\epsilon\vec{r}\cdot[\partial^2V_n(\vec{I}_r)/\partial\vec{I}^2]\cdot\vec{r}$, $\vec{n}\in\mathcal{G}$. In the opposite case (for instance if H_0 is linear in $\vec{I}$) the method of Fukuyama *et al.*[8] can be used. It amounts to going to the action-angle variables of the integrable Hamiltonian $H_0(\vec{I})+\epsilon V_r(\vec{I})\cos(\vec{r}\cdot\vec{\theta}+\chi_r)$. The interaction of the primary resonances of the transformed Hamiltonian yields again a steep rise of the width of the stochastic layer.[8]

We illustrate the case of a unique primary resonance on H_w. Here $\vec{r}=(2,-2)$ and $\vec{m}=(3,-2)$. There results $a=24$, $k=\frac{5}{4}$, $\beta_m=-\sqrt{2}/(2\omega_r)$; $\vec{I}_r$

corresponds to $I_1=5I_2$ and can be computed from $H_0(\vec{I}_r)=E$. A straightforward application of the above formulas yields $E_w'=0.078$ for the LSS threshold estimate.

We now deal with $h^\pm$. The role of ϵ is played here by $\sqrt{E}$, where E is the total energy (this is obvious by replacing the x_i and y_i by the $x_i/\sqrt{E}$ and $y_i/\sqrt{E}$). In order to give $h^\pm$ the structure of Eq. (1), we need to go to some action-angle variables. We can do this in two ways. The first one consists in taking the action-angle variable $(\vec{I},\vec{\theta})$ of the quadratic part of $h^\pm$ defined by $x_i=(2I_i)^{1/2}\cos\theta_i$, $y_i=(2I_i)^{1/2}\sin\theta_i$, $i=1,2$. There results $H_0^\pm(\vec{I})=I_1+I_2$, $\mathcal{I}=\{(0,1),(0,3),(2,-1),(2,1)\}$, $V_{0,1}^\pm=(2I_2)^{1/2}(I_1\pm I_2)/2$, $V_{0,3}^\pm=\pm\sqrt{2}I_2^{3/2}/6$, $V_{2,1}^\pm=V_{2,-1}^\pm=(2I_2)^{1/2}I_1/2$. The Hamiltonian $H^\pm$ so obtained has no primary resonance. When the Kolmogorov transformation is applied once, one gets

$$H_0'^-(\vec{J})=J_1+J_2-(5J_1^2+5J_2^2-4J_1J_2)/12,$$

$$H_0'^+(\vec{J})=J_1+J_2-5(J_1^2+J_2^2+4J_1J_2)/12.$$

There is now a unique resonant term $\vec{r}=(2,-2)$ with amplitude $W_r^-=-7J_1J_2/6$ or $W_r^+=\frac{5}{7}W_r^-$. As $\vec{J}_r=(\frac{1}{2}\sqrt{E},\frac{1}{2}\sqrt{E})$, there results $|U_r^-|=7E/24$, $|U_r^+|=5E/24$, $a^-=-28E/3$, $a^+=20E/3$. The closest nonresonant term is of order $\epsilon^3=E^{3/2}$ and has the direction $\vec{m}=(4,-5)$ so that $k=\alpha_m=\frac{9}{4}$ and $\beta_m=\frac{1}{2}$; the parameter λ of Eq. (5) is 5. This yields the estimates $E_t^-=0.08$ and $E_t^+=0.115$.

A second way of dealing with $h^\pm$ is to use first the generalized Birkhoff normal forms.[9] We perform on $h^\pm$ two successive canonical transformations with generating functions

$$f_1^\pm(\vec{x},\vec{\eta})=x_1\eta_1+x_2\eta_2+\tfrac{1}{3}(\pm 2\eta_2^3/3+2\eta_1^2\eta_2\pm x_2^2\eta_2+2x_1x_2\eta_1+x_1^2\eta_2),$$

and

$$f_2^\pm(\vec{\xi},\vec{y}')=\xi_1y_1'+\xi_2y_2'+\{17[\xi_1y_1'^3+\xi_2y_2'^3+K^\pm(\xi_1y_1'y_2'^2+\xi_2y_1'^2y_2')]+7[\xi_1^3y_1'+\xi_2^3y_2'+K^\pm(\xi_1\xi_2^2y_1'+\xi_1^2\xi_2y_2')]\}/144,$$

where $K^+=3$ and $K^-=1$, that define successively the variables $(\vec{\xi},\vec{\eta})$ and $(\vec{x}',\vec{y}')$. They yield the new Hamiltonian

$$h^{\pm\prime}(\vec{x}',\vec{y}')=\tfrac{1}{2}(x_1'^2+x_2'^2+y_1'^2+y_1'^2)+h_2^\pm+h_3^\pm+O(E^2),$$

where

$$h_2^\pm(\vec{x}',\vec{y}')=-[5(x_1'^2+x_2'^2+y_1'^2+y_2'^2)^2+A^\pm]/48$$

and

$$h_3^\pm(\vec{x}',\vec{y}')=2\{2(x_1'x_2'+2y_1'y_2')B_{12}^\pm+[x_1'^2\pm x_2'^2+2(y_1'^2\pm y_2'^2)]B_{21}^\pm\}/27,$$

and

$$A^-=-28(x_1'y_2'-x_2'y_1')^2,\quad A^+=20(x_1'x_2'+y_1'y_2'),$$

$$B_{ij}^-=x_i'^3+x_i'x_j'^2-4x_i'y_j'^2+4x_j'y_i'y_j',$$

and $B_{ij}^+=x_i'^3+3x_i'x_j'^2$. Action-angle variables are defined for $h^{\pm\prime}$ as they were before for $h^\pm$ and yield a Hamiltonian with an angle-independent part $H_0'^\pm$ already defined and a set $\mathcal{I}^\pm$ of angle-dependent terms. Among these terms, only the $(2,-2)$ one is of order $\epsilon^2=E$ and its amplitude is again given by $W_r^\pm$. It is also the only one to be resonant so that $\vec{r}=(2,-2)$. We again use the method for one primary resonance. Among the other angle-dependent terms which all are of order $\epsilon^3=E^{3/2}$, the direction closest to $\vec{r}$ is $\vec{m}=(2,-3)$ [also $\vec{m}'=(3,-2)$ for h^+]. Both $\vec{m}$ and $\vec{m}'$ yield $\alpha_m=\frac{5}{4}$ and $|\beta_m|=\frac{1}{2}$. The parameter λ of Eq. (5) is 3; parameters $\vec{J}_r$, $a^\pm$, and $|U_r^\pm|$ take the same values as before. This yields the estimates $E_t^{-\prime}=0.106$ and $E_t^{+\prime}=0.15$. The fact that there is only one angle-dependent term of order $\epsilon^2=E$ explains why $E_t^{-\prime}$ is better than E_t^- since we are not led to neglect angle-dependent terms of order $\epsilon^2=E$ as before. Going to higher-order normal forms would not improve $E_t^{-\prime}$ since there will still remain the error due to forming the Taylor expansion of H in order to get H_p.

The analytical methods of this Letter and those already available[1-4] for computing the threshold of LSS show the usefulness of the structure of Eq. (1) which appears as a standard form for this computation. All these methods approximate the considered Hamiltonian by a new one with only two cosine terms in Eq. (1). Further investigation is left to be done in order to avoid this approximation or to give error bars on the estimates.

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Stochastic Instability of Sine-Gordon Solitons

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A breather soliton of the sine-Gordon equation is shown to behave stochastically in the presence of an external oscillating field, which leads to random pair creations of kink and antikink solitons.

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Recently there has been a surge of interest in statistics of solitons in various physical fields.[1,2] Since nonlinear wave equations governing solitons are completely integrable, there must be some mechanisms which enable us to consider a statistical ensemble of solitons. For nontopological solitons the thermal mechanism may be natural since its activation energy can be very small. But for phase solitons such as kinks of the sine-Gordon (SG) equation, which have a finite creation energy, nonthermal excitation mechanisms will play an important role, especially in nonequilibrium systems. In this Letter, I show that kink solitons of the SG equation can be randomly created through the stochastic instability of breathers in the presence of an oscillating external field. This system is the model of many physically interesting phenomena such as charge-density waves in a one-dimensional condensate in the presence of an ac electric field,[3] the magnetic-flux propagation on a Josephson-junction transmission line with an ac external current,[4] and the creation of baryons from a meson through the interaction with an external oscillating field.[5]

The interaction between SG solitons and an external field is described by

$$u_{tt} - u_{xx} + \sin u = \epsilon \cos \omega t \equiv f(t), \tag{1}$$

where ϵ is a small parameter and $f(t)$ is an external field. The Hamiltonian of Eq. (1) is

$$H = \tfrac{1}{2}\int dx[u_t^2 + u_x^2 + 2(1-\cos u)] - f\int u\,dx, \tag{2}$$

while the total momentum of the field $P \equiv -\int u_x u_t \times dx$ is still conserved when $u(\infty, t) - u(-\infty, t) = 0$ which is satisfied for a breather and a kink-antikink pair. Therefore we can choose a reference frame such that the center of mass of a breather (or a kink-antikink pair) is at rest even in the presence of the external field. This is equivalent to eliminating one degree of freedom of a breather and enables us to perform a simple perturbational approach.

If $\epsilon = 0$, Eq. (1) is completely integrable in terms of canonical variables which consist of the scattering data of the associated linear eigenvalue equation,[6] and it has the following soliton solution:

$$u = 4\arctan[T(t)\operatorname{sech}2k(x - x_0)], \tag{3}$$

where x_0 is constant and $T(t) = \sin(\alpha/16)\tan\theta$, $k = \frac{1}{2}\sin\theta$ for a breather and $T(t) = (e^p + e^{-p})\sinh(q/16)/(e^p - e^{-p})$, $k = e^p + e^{-p}$ for a kink-antikink pair. Here (θ, α) and (p, q) are the canonical conjugate variables which are given by

$$\theta(t) = \theta(0), \quad p(t) = p(0),$$

$$\alpha(t) = 16t\cos\theta + \alpha(0), \quad q = 32(e^p - e^{-p})t + q(0),$$

where the unperturbed Hamiltonian is given by $H_0 = 16\sin\theta$ or $H_0 = 32(e^p + e^{-p})$. Since the long-time behavior of θ, p is of interest in the presence of the external oscillating field, the averaging method for Hamiltonian systems can be applied. Substituting Eq. (3) into Eq. (2), we have the approximate Hamiltonian

$$H = H_0 + (2\pi f/k)\ln[-T + (1 + T^2)^{1/2}], \tag{4}$$