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Superfluidity of a Spin-$\frac{1}{2}$ Bose Fluid: Spin-Polarized Hydrogen

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It is shown here that superfluids (such as ^{3}He-*A* and Bose-condensed spin-polarized hydrogen) whose order parameters transform like an angular momentum eigenstate $|j, m\neq 0\rangle$ will have similar superfluidity—similar Josephson and vorticity equations. In particular, if spin-polarized hydrogen condenses into the *b* state as currently believed, it will not have a stable superflow. Nonuniform magnetic fields can also induce in it a persistent current similar to that of ^{3}He-*A*.

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The recent stabilization of atomic spin-polarized hydrogen[1] (SPH) has generated much hope in the observation of Bose-Einstein condensation in this system. A hydrogen (H) atom can be regarded as a boson because it contains two fermions. Recently, Siggia and Ruckenstein[2] pointed out that SPH, when condensed, will behave like a spin-$\frac{1}{2}$ Bose superfluid. This is because at low temperatures, only the lowest two hyperfine states, usually refered to as *a* and *b*, are important. They correspond to the alignment and misalignment of the proton spin with the external field, modified by the hyperfine interaction. The electron spin is basically held fixed by the external field because it has a magnetic moment (μ_e) 10^3 times larger than the proton's (μ_p). More recently, Statt and Berlinsky[3] pointed out that because of molecular recombination effects, condensation is mostly likely to occur in the *b* state.

The purpose of this paper is to show that, nc matter which hyperfine state SPH condenses into, the flow properties of this prospective $S=\frac{1}{2}$ Bose superfluid will be very similar to those of ^{3}He-*A*, a *p*-wave BCS superfluid well known for its peculiar superfluidity. In fact, we shall see that all superfluids whose order parameter transforms like an angular momentum eigenstate $|j, m\neq 0\rangle$ will have similar flow properties.

^{3}He-*A* consists of Cooper pairs which are orbital angular momentum eigenstates $|L=1, \vec{L}\cdot\hat{l}=1\rangle$ along a certain direction $\hat{l}$. The pair wave function is $Y_{11}(\vec{\rho})\propto\hat{\phi}\cdot\vec{\rho}$, where $\hat{\phi}=\hat{\phi}_1+i\hat{\phi}_2$, $\hat{\phi}_1\cdot\hat{\phi}_2=0$, $\hat{l}=\hat{\phi}_1\times\hat{\phi}_2$, and $\vec{\rho}$ is the relative displacement of the helium atoms in the pair. The triad structure of the orbital order parameter $\hat{\phi}$ is the cause of the peculiar flow properties of ^{3}He-*A*. For example, in simply connected containers (because of the surface boundary condition on $\hat{l}$), ^{3}He-*A* will carry a persistent current whose magnitude is of the order of a few vortices.[4] On the other hand, superflows (with larger winding numbers) in multiply connected containers are not intrinsically stable.[5] All these are very different from the behavior of the more familiar superfluids such as ^{4}He and superconductors, whose order parameters are scalars.

That SPH can be treated as a $S=\frac{1}{2}$ Bose gas can be formulated as follows. The Hamiltonian of SPH is $H=T+Z+V$, where T is the kinetic energy, V is the interaction between the H atoms, and Z is the sum of the Zeeman energy and the hyperfine interaction (g),

$$Z=\int d^3x\,\hat{\eta}^\dagger[\tfrac{1}{2}(\mu_e\vec{\sigma}-\mu_p\vec{\tau})\cdot\vec{H}_{\rm ext}+\tfrac{1}{4}g\vec{\tau}\cdot\vec{\sigma}]\hat{\eta}.$$

Here, $\vec{\tau}$ ($\vec{\sigma}$) is the proton (electron) Pauli matrix, $\vec{H}_{\rm ext}(\vec{r})$ is the external field, and $\hat{\eta}=(\hat{\eta}_{\mu i})$ is the hydrogen Bose operator with electron (proton) spin index μ (i). It is straightforward to show that the matrix $U=\exp[i\vec{\tau}\times\vec{\sigma}\cdot\hat{n}\epsilon/4]$, with $\epsilon=\cot^{-1}[(\mu_p+\mu_e)H_{\rm ext}/g]$ and $\hat{n}=\hat{H}_{\rm ext}$, diagonalizes Z locally, so that

$$Z=\int d^3x\,\hat{\eta}^\dagger U^\dagger[\tfrac{1}{2}(\overline{\mu}_e\vec{\sigma}-\overline{\mu}_p\vec{\tau})\cdot\vec{H}_{\rm ext}+\tfrac{1}{4}g(\vec{\tau}\cdot\hat{n})(\vec{\sigma}\cdot\hat{n})]U\hat{\eta},$$

where

$$\bar{\mu}_p - \mu_p = \bar{\mu}_e - \mu_e = (g/2H_{\rm ext})\tan\epsilon/2.$$

In the uniform case, $\hat{n}(\vec{r}) = \hat{z}$, the lowest two states of Z are $(U\hat{\eta})_{-1i}$, $i = \pm 1$. They are the so-called a and b states, and are far away from other excited states because $\mu_e/\mu_p \sim 10^3$. When the direction of the external field $\hat{n}$ is slowly varying, the dynamics of the low-lying states can be described by approximating $U\hat{\eta}$ as $(U\hat{\eta})_{\mu i} = f_\mu \hat{\psi}_i$, where f is a normalized *static* electron spinor fixed by the external field, $\hat{n}\cdot\vec{\sigma}f = -f$, $f^\dagger f = 1$. In terms of $\hat{\psi}$, which obeys Bose commutation relations, H assumes the form of an interacting spin-$\frac{1}{2}$ Bose gas. To the first order in ϵ ($\epsilon \sim 10^{-3}$ for $H_{\rm ext} = 100$ kG), we have

$$H = \int d^3x \{(\hbar^2/2M)|(\nabla + i\vec{A})\hat{\psi}|^2 - \mu_p\vec{\Delta}\cdot\hat{\psi}^\dagger\vec{\tau}\hat{\psi}/2 - \hat{\psi}^\dagger\hat{\psi}[\mu_e H_{\rm ext} - (\hbar^2/8M)(\nabla_i n_j)^2]\} + \tfrac{1}{2}\int d^3x\, d^3y\, V(\vec{x},\vec{y})\hat{\psi}_i^\dagger(\vec{x})\hat{\psi}_j^\dagger(\vec{y})\hat{\psi}_j(\vec{y})\hat{\psi}_i(\vec{x}), \tag{1}$$

where $A_i = (M/\hbar)w_i - \nabla_i\hat{n}\times\hat{n}\cdot\vec{\tau}\epsilon/4$, $\vec{w} = (\hbar/Mi)f^\dagger\nabla f$, and

$$\mu_p\vec{\Delta} = \hat{n}[\bar{\mu}_p H_{\rm ext} + g/2 + \epsilon(\nabla_i n_j)^2\hbar^2/4M] - (\nabla_i\epsilon\nabla_i\hat{n})\hbar^2/4M.$$

The order parameter of (1) is a spinor $\Psi = \langle\hat{\psi}\rangle$. In the literature, Ψ is usually represented in component form $\Psi^\dagger = (\Psi_a^*, \Psi_b^*)$ with respect to the external field. In a uniform field, the question of whether only one or both components of Ψ would be formed has been studied.[2,3] By noting that any spinor Ψ is a maximum spin state along a certain direction $\hat{l}$, $\hat{l}\cdot\vec{\tau}\Psi = \Psi$, the question of how many components there are in the condensate is simply a question of whether $\hat{l}$ is aligned with $\hat{n}$ (one component if $\hat{l} = \pm\hat{n}$, and two otherwise). A "two-component" superfluid with respect to a given axis can be a "single component" to a different axis, and vice versa.

In thermodynamic equilibrium, one expects $\hat{l}(\vec{r}) \simeq \hat{n}(\vec{r})$, i.e., condensation takes place in the local a state. There is, however, the realistic question of whether SPH could reach the true equilibrium state within the time scale of the experiment. It is believed that because of the molecular recombination effect caused by the hyperfine mixing in the a state, most of the atoms in the normal phase will be in the b state.[3] One therefore expects to find $\hat{l} = -\hat{n}$ in the condensed phase. While it is important to investigate the effects of the field gradients on the recombination rate and the magnetic relaxation time, which determine how fast the system reaches equilibrium, it is clear that whenever condensation occurs, a spinor field Ψ, and hence a vector field $\hat{l}$, will result. Here, I focus on those phenomena caused by the nonuniformity of $\hat{l}$, as generated by external fields, or by hydrodynamic effects.

The relation between ^{3}He-A and SPH is that their order parameters transform like an angular momentum eigenstate $\Psi^{(jm)}$, $m \neq 0$. That is to say, the normalized order parameter $\zeta \equiv \Psi/|\Psi|$ (omitting the superscript jm) can be obtained from a reference order parameter ζ_0 ($J_z\zeta_0 = m\zeta_0$, $\zeta_0^\dagger\zeta_0 = 1$) by a rotation, $\zeta = \exp(-i\theta\hat{\nu}\cdot\vec{J})\zeta_0$, where $\vec{J}$ is the angular momentum operator. This also implies that $\hat{l}\cdot\vec{J}\zeta = m\zeta$, where $\hat{l} = R(\hat{\nu},\theta)\hat{z}$, and $R(\hat{\nu},\theta)$ is a rotation about the axis $\hat{\nu}$ through an angle θ. [For ^{3}He-A, $j = m = 1$, we have $\vec{J} = -i\vec{\rho}\times\nabla\rho$, $\zeta_0 = \frac{3}{2}^{1/2}(\hat{x} + i\hat{y})\cdot\hat{\rho}$, $\hat{\rho} = \vec{\rho}/\rho$, and $\zeta^\dagger\zeta \equiv \int|\zeta|^2 d\hat{\rho}/4\pi$. For SPH, we have $j = m = \frac{1}{2}$, $\vec{J} = \vec{\tau}/2$, and $\zeta_0^\dagger = (1, 0)$.]

Since normalized order parameters at neighboring points must be related by infinitesimal rotations, we have $\delta\vec{r}\cdot\nabla\zeta = i\delta r_i\Omega_{ij}J_j\zeta$, where Ω_{ij} is a real tensor. Under a Galilean transformation, $\vec{r} \to \vec{r} + \vec{u}t$, ζ acquires a phase factor $\exp(iM\vec{u}\cdot\vec{r}/\hbar)$, where M is the mass of the boson or the Cooper pair. The quantity

$$(v_s)_i \equiv (\hbar/Mi)\zeta^\dagger\nabla_i\zeta = (m\hbar/M)\Omega_{ij}l_j$$

therefore transforms like a velocity.[6] It is easy to show that $\Omega_{ij} = (M/m\hbar)v_{si}l_j + (\nabla_i\hat{l}\times\hat{l})_j$. The spatial variations of ζ are thus completely specified by $\vec{v}_s$ and $\nabla_i\hat{l}_j$. Likewise, the supercurrent and free energy generated by $\nabla\zeta$ can be written entirely in terms of these quantities. According to Ref. 4, because of the condition $\nabla_i\nabla_j\zeta = \nabla_j\nabla_i\zeta$, $\vec{v}_s$ and $\hat{l}$ are further related (for all $j, m \neq 0$) by

$$\vec{\nabla}\times\vec{v}_s = (m\hbar/M)\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}l_\alpha\nabla l_\beta\times\nabla l_\gamma. \tag{2}$$

Before discussing the implications of (2) on SPH, for simplicity and for theoretical reasons, I shall first consider the following $S = \frac{1}{2}$ Bose gas: $H_{1/2} = (\hbar^2/2M)\int|\nabla\hat{\psi}|^2 - \frac{1}{2}\int\hat{\psi}^\dagger\vec{\tau}\hat{\psi}\cdot\mu_p\vec{\Delta} + V$. Its relation with SPH will be discussed shortly. Equation (2) implies the following for $H_{1/2}$:

(i) Persistent current and macroscopic angular momentum.—When $\hat{l}(\vec{r})$ is parallel to $\hat{n}(\vec{r})$, the supercurrent of $H_{1/2}$ is of the form[7] $\vec{g} = \rho_s\vec{v}_s$ [see

also discussion (a) below]. Equation (2) implies that a nonuniform $\hat{l}$ will produce a current $\vec{g}$ and an angular momentum $\vec{L} = \int \vec{r} \times \vec{g}$ whose magnitude is similar to that of a single vortex.[4] The persistent current is a remarkable effect of Bose condensation. It occurs even in the case of an ideal Bose gas. The reason is that as long as $\hat{n}$ is varying, there are no spinor eigenstates with real arguments. The ground state is genuinely complex and carries a current, which will be magnified enormously by condensation.

(ii) Stability of superflow.—For simplicity, consider the case $\hat{n} = \vec{H}_{ext} = \hat{z}$. The Ginzburg-Landau free energy of $H_{1/2}$ is $F = F_B(|\Psi|^2, \hat{l}\cdot\hat{z}) + \frac{1}{2}K|\nabla\Psi|^2$, where F_B is the bulk free energy which will be minimized when $\hat{l} = \hat{z}$. The gradient energy can be rewritten as

$$F_G = \tfrac{1}{2}K\{[(2M/\hbar)^2 v_s^{\,2} + (\nabla_i l_j)^2]|\Psi|^2 + (\nabla|\Psi|)^2\}.$$

Equation (2) allows transfer of energy between the superflow v_s and $\nabla_i l_j$, and transfer will occur if either type of energy is too high.

Let us consider the simplest flow configuration: $\hat{l} = \hat{z}$, $\vec{v}_s = \vec{u}$. To test its stability,[5] we consider small variations of $\hat{l}$ (up to second order) of the form $\delta\hat{l} = \vec{\lambda} - \frac{1}{2}\lambda^2\hat{z}$, $\vec{\lambda}\cdot\hat{z} = 0$, with an accompanying change in $\vec{v}_s$,

$$(2M/\hbar)\delta\vec{v}_s = \tfrac{1}{2}(\lambda_x\nabla\lambda_y - \lambda_y\nabla\lambda_x) + \nabla\varphi,$$

because of (2). The change in the free energy (after minimizing with respect to $\vec{\nabla}\varphi$) is

$$\delta F = \tfrac{1}{2}K[(\nabla_i\lambda_j)^2 + (2M\vec{u}/\hbar)(\lambda_x\nabla\lambda_y - \lambda_y\nabla\lambda_x)]\cdot|\Psi|^2 - \tfrac{1}{2}\vec{M}\cdot\vec{\Delta}\lambda^2,$$

where the last term comes from the variation of the bulk free energy, and $\vec{M}$ is the magnetization of the initial configuration. The gradient energy is always unstable.[8] The most unstable mode is a helical distortion, $\vec{\lambda} = \lambda R(\hat{z}, \vec{q}\cdot\vec{r})x$, $\vec{q} = -M\vec{u}/\hbar$. Collapse of flow will take place if the flow energy is larger than the field energy. On the other hand, if the initial configuration is the b state, $\hat{l} = -\hat{z}$, then there will be no energy barrier preventing the decay of flow through $\hat{l}$ distortions.[9]

(iii) Josephson equation.—The time derivative of $\vec{v}_s$ gives

$$\partial_t\vec{v}_s = \frac{\hbar}{Mi}\nabla(\zeta^\dagger\partial_t\zeta) + \frac{\hbar}{2M}\epsilon_{\alpha\beta\gamma}l_\alpha\partial_t l_\beta\nabla l_\gamma. \qquad (3)$$

From the equilibrium motion of ζ, the first term can be identified as $\nabla[-\mu + \hat{l}\cdot(\vec{\Delta} - \vec{M}/\chi)]$ in the nondissipative hydrodynamic limit, where μ is the chemical potential, $\vec{M}$ is the magnetization, and χ is the susceptibility. Thus, chemical potential gradients or nonuniform nonequilibrium magnetization can both drive $\hat{l}$ and $\vec{v}_s$ in motion.

(iv) Line defects.—If $\vec{\Delta} = 0$ in $H_{1/2}$, the order parameter space is precisely SU(2). Since $\Pi_1(\mathrm{SU}(2)) = 0$, there are no topologically stable defects.[10] The effect of $\vec{\Delta}$ is to stabilize certain kinds of defects by producing an energy barrier in the process of deformation. Although $\vec{\Delta}$ tends to stabilize the vortices in the a state, the vortices in the b state, $\hat{l} = -\hat{z}$, remain energetically unstable, as can be seen from the energetics of the family, $\zeta^\dagger(t) = (\sin(t\pi/2), e^{-im\varphi}\cos(t\pi/2))$, $0 \le t \le 1$, which implies $dF(t)/dt < 0$ when $\hat{n}(\vec{r}) = \hat{z}$.

Returning to SPH, I note the following:

(a) Strictly speaking, the order parameter of SPH is the tensor $\langle\eta_{\mu i}\rangle = U^\dagger f\Psi$ rather than simply the spinor $\Psi = |\Psi|\zeta$; although only Ψ itself contains dynamics. Since

$$\nabla_i(f\zeta) = i[(M/h)(v_s + w)_i + \nabla_i\hat{l}\times\hat{l}\cdot\vec{\tau}/2 + \nabla_i\hat{n}\times\hat{n}\cdot\vec{\sigma}/2]f\zeta,$$

the gradients of the tensor $f\zeta$ are specified by $\nabla_i l_j$, $\nabla_i n_j$, and the sum $\vec{v}_s + \vec{w}$. The general form of the supercurrent allowed by symmetry is therefore[11]

$$g_i = \rho_s(v_s + w)_i + \hat{n}\times\hat{l}\cdot(B\nabla_i\hat{n} + C\nabla_i\hat{l}).$$

It is expected from Eq. (1) that $B \sim \epsilon$, and C is related to the interaction of the H atoms.[12] Since $\vec{w}$ and $-\vec{n}$ are the "velocity" and the spin quantization axis of f (similar to $\vec{v}_s$ and $\hat{l}$ of ζ), they therefore satisfy Eq. (2) with the replacement $\vec{v}_s \to \vec{w}$, $\hat{l} \to -\hat{n}$. For $\hat{l} = +(-)\hat{n}$ [so that $f\zeta$ becomes a spin-0 (−1) object], we have $\nabla\times\vec{v}_s = -(+)\nabla\times\vec{w}$. A nonuniform field $\hat{n}$ will therefore generate a persistent current as discussed in (i) as long as the condensate is not in the local a state. On the other hand, whenever $\hat{l}$ moves away from $\hat{n}$, the hydrodynamics of SPH must be described by both (2) and (3).

(b) When the external field is uniform, $\hat{n} = \hat{z}$, Eq. (1) reduces to $H_{1/2}$. Discussion (ii) and the example $\zeta^\dagger(t)$ in (iv) therefore apply to SPH. Since $\mu_p\Delta \sim 50$ mK when $H_{ext} = 100$ kG, the external field strongly stabilizes the a state (requiring a critical velocity of 10^3 cm/sec). On the other hand, superflows in the b state remain unstable. One therefore expects that in a ρ_s

measurement (such as a study of the oscillation of a cylindrical cavity containing SPH), the b state will show a much smaller effective superfluid density and will produce a lot more damping than the a state.

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[6]The representation $\zeta = \exp(-i\theta\hat{\nu}\cdot\vec{J})\zeta_0$ also implies that $v_{si} = (m\hbar/M)\varphi_1\cdot\nabla_i\hat{\varphi}_2$ for all j, $m\neq 0$, where $\hat{\varphi}_1 + i\hat{\varphi}_2 = R(\hat{\nu},\theta)(\hat{x}+i\hat{y})$. The linearized hydrodynamics of the $S=\frac{1}{2}$ Bose fluid is discussed by B. I. Halperin, Phys. Rev. B 11, 178 (1975). The present definition of v_s reduces to the one therein when $\hat{l}$ becomes uniform.

[7]In ^{3}He-A, $\vec{g}$ contains terms proportional to $\nabla\times\hat{l}$. These terms are absent here because of the rotation symmetry in spin space.

[8]Away from the Ginzburg-Landau region, symmetry requires that, when $\hat{n}=\hat{z}$

$$F_G = \tfrac{1}{2}\rho_s v_s^2 + \tfrac{1}{2}\tilde{K}(\nabla_i\hat{l}_j)^2 + C\hat{z}\times\hat{l}\cdot(\vec{v}_s\cdot\nabla)\hat{l},$$

but places no constraints on the coefficients. The last term will produce a stronger (linear) instability.

[9]The family $\zeta^+(\vec{r},t) = (\sin\pi t/2,\ \exp(-iM\vec{u}\cdot\vec{r}/\hbar)\cos(\pi t/2))$ reduces the energy $F_B + F_G$ monotonically as $(\frac{1}{2}K|\Psi|^2u^2 + M\Delta)\cos^2(\pi t/2) - M\Delta$, as t varies from 0 to 1.

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[11]The expression of $\nabla(f\zeta)$ indicates that only the sum $\vec{v}_s + \vec{w}$ is relevant. The current $\vec{g}$ does not depend on $\vec{v}_s$ and $\vec{w}$ separately. Since f is static, all the dynamics can be absorbed in $\vec{v}_s$, with $\vec{w}$ treated as a static background.

[12]For the ideal gas ($H_{1/2}$ with $V=0$) it can be shown that $C=0$ at $T=0$.

Tortuosity and Acoustic Slow Waves

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The experimental measurements of tortuosity of porous structures using either the acoustic index of refraction of superfluid ^{4}He or the electrical conductivity are shown to agree with each other. This and other measured parameters are used to calculate directly the acoustic speeds of water-saturated, fused-glass-bead samples; there are no adjustable parameters and agreement with experiment is excellent. The dependence of tortuosity on pore volume fraction, φ, is discussed.

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In this Letter we consider the acoustic and electrical properties of porous, fluid-saturated, fused-glass-bead samples (Ridgefield Sandstone) which have the unusual property that they support two distinct longitudinal acoustic modes.[1] The class of porous materials being considered is characterized by the unique topological property that the fluid and solid components each forms its