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Position-Momentum Uncertainty Relations in Stochastic Mechanics

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A simple inequality relating the root mean square deviations of position and osmotic velocity for a diffusion process is presented and, in the framework of Nelson's stochastic mechanics, is related to the Heisenberg position-momentum uncertainty relations.

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In this paper we provide a simple derivation of the Heisenberg position-momentum uncertainty relations in the framework of Nelson's stochastic mechanics.[1] In such a scheme, which associates a diffusion process to every given quantum state, we show that the uncertainty relations come from a purely kinematical fact about classical diffusion, which can be traced back to the nondifferentiability of the typical sample path.

We refer the reader to Ref. 2 for a deep and extensive review of the interplay between quantum mechanics and the theory of stochastic processes. Here we just give a simple introduction of the probabilistic terms used in our discussion.

Consider a particle of mass m, whose motion on the real line is described by the stochastic differential equation

$$dq(t) = b_+(q(t), t)\,dt + dw(t) \quad (dt > 0). \tag{1}$$

Equation (1) describes a random disturbance $dw(t)$ [supposed here to be Gaussian, with expectation $E(dw(t)) = 0$ and variance $E(dw(t)^2) = 2\nu\, dt$, where ν is called the diffusion coefficient] superimposed on the otherwise deterministic evolution determined by the velocity field $b_+(x, t)$. We recall, first of all, that what would seem the most natural definition of momentum to be associated with the random motion of the particle, namely $m\, dq/dt$, is in fact incorrect, because of the fact that $dw = O(dt^{1/2})$ prevents the existence of the relevant limit. Two alternative definitions were proposed in Ref. 1: The first (i) is

$$p_+(t) = mv_+(q(t), t), \tag{2}$$

where the mean forward velocity field $v_+(x, t)$ is defined by

$$v_+(x, t) = \lim_{\Delta t \to 0^+} E\left(\frac{q(t+\Delta t) - q(t)}{\Delta t}\bigg|_{q(t)=x}\right). \tag{3}$$

The operational meaning of the conditional expectation $E(\cdots)$ appearing in the definition is clear: $v_+(x, t)$ is the mean slope with which those sample paths that at time t are in x *leave* x. It is not difficult to check that

$$v_+(x, t) = b_+(x, t). \tag{4}$$

The second definition (ii) is

$$p_-(t) = mv_-(q(t), t), \tag{5}$$

where the mean backward velocity field $v_-(x, t)$ is defined by

$$v_-(x, t) = \lim_{\Delta t \to 0^+} E\left(\frac{q(t) - q(t-\Delta t)}{\Delta t}\bigg|_{q(t)=x}\right). \tag{6}$$

Operationally, $v_-(x, t)$ is the mean slope with which those sample paths that at time t are in x *enter* x.

Definitions (i) and (ii), though in principle distinct, are not completely independent. They are indeed related to the probability density $\rho(x,t)$ of the position of the particle by the kinematical relation

$$v_+(x,t) - v_-(x,t) = 2\nu \frac{1}{\rho(x,t)} \frac{\partial \rho(x,t)}{\partial x} . \tag{7}$$

Multiplying both sides of (7) by $m\rho$ and integrating over x, we obtain

$$E(p_+(t)) = E(p_-(t)) . \tag{8}$$

Namely, the two random variables $p_\pm(t)$ have the same expectation. Multiplying both sides of (7) by $m\rho x$, integrating over x, and using (8), we obtain

$$E\left([q(t) - E(q(t))]\left[\frac{p_-(t) - p_+(t)}{2} - E\left(\frac{p_-(t) - p_+(t)}{2}\right)\right]\right) = m\nu . \tag{9}$$

Schwarz's inequality then implies

$$\Delta q \, \Delta \delta p \geq m\nu , \tag{10}$$

where Δq and $\Delta \delta p$ are, respectively, the root mean square deviations of the random variables $q(t)$ and $\delta p(t) = [p_+(t) - p_-(t)]/2$. Our main point is that inequality (10) is not merely an analog[3] to the Heisenberg position-momentum uncertainty relations but, indeed, in the framework of Nelson's stochastic mechanics, implies and, we hope, clarifies them.

To a quantum state of a particle of mass m described by a normalized wave function $\psi(x,t) = \exp[R(x,t) + iS(x,t)/\hbar]$ stochastic mechanics associates a stochastic process with diffusion coefficient

$$\nu = \hbar/2m , \tag{11}$$

whose probability density is given by

$$\rho(x,t) = |\psi(x,t)|^2 , \tag{12}$$

and whose current velocity is given by

$$v(x,t) = \frac{v_+(x,t) + v_-(x,t)}{2} = \frac{1}{m} \frac{\partial S(x,t)}{\partial x} . \tag{13}$$

Because of (11) and (12), for the process $q(t)$ associated with the wave function $\psi(x,t)$, Eq. (7) specializes to the following expression for the osmotic velocity:

$$u(x,t) = \frac{v_+(x,t) - v_-(x,t)}{2} = \frac{\hbar}{m} \frac{\partial R(x,t)}{\partial x} . \tag{14}$$

Inequality (10) implies then that the root mean square deviations of the random variables $q(t)$ and $u(q(t),t)$ satisfy

$$\Delta q \, \Delta m u \geq \hbar/2 . \tag{15}$$

In order to connect this inequality with the position-momentum uncertainty relations, we observe that, because of (12), we have in fact

$$\Delta q = \{E([q(t) - E(q(t))]^2)\}^{1/2} = [\langle \psi(t), x^2 \psi(t)\rangle - \langle \psi(t), x\psi(t)\rangle^2]^{1/2} . \tag{16}$$

Next, we observe that the relevant expectation of the momentum operator $p = (\hbar/i)\partial/\partial x$ in

$$\Delta p = [\langle \psi(t), p^2 \psi(t)\rangle - \langle \psi(t), p\psi(t)\rangle^2]^{1/2} \tag{17}$$

can be easily read from the mean stochastic velocities of the process $q(t)$. We have, indeed,

$$E(p_+(t)) = E(p_-(t)) = \int \rho(x,t) \frac{\partial S(x,t)}{\partial x} dx = \langle \psi(t), p\psi(t)\rangle , \tag{18}$$

and, similarly,

$$E(p_+^2(t)) = \langle \psi, p^2 \psi\rangle + 2\hbar \int \rho(x,t) \frac{\partial S(x,t)}{\partial x} \frac{\partial R(x,t)}{\partial x} dx . \tag{19}$$

Namely,

$$E(p_+^2(t) + p_-^2(t))/2 = \langle \psi(t), p^2 \psi(t)\rangle . \tag{20}$$

We have, therefore,

$$\Delta p = [\langle \psi(t), p^2\psi(t)\rangle - \langle \psi(t), p\psi(t)\rangle^2]^{1/2} = \left(\frac{E(p_+^2(t)) - [E(p_+(t))]^2}{2} + \frac{E(p_-^2(t)) - [E(p_-(t))]^2}{2}\right)^{1/2} . \tag{21}$$

Equation (21) is to be compared with

$$\Delta m u = \left[E\left(\left(\frac{p_+(t) - p_-(t)}{2}\right)^2\right)\right]^{1/2} . \tag{22}$$

Since

$$(\Delta p)^2 - (\Delta mu)^2 = E\left(\left(\frac{p_+(t)+p_-(t)}{2}\right)^2\right) - E(p_+(t))E(p_-(t))$$
$$= \int dx\, \rho(x,t)\left(\frac{\partial S(x,t)}{\partial x}\right)^2 - \left(\int dx\, \rho(x,t)\frac{\partial S(x,t)}{\partial x}\right)^2, \tag{23}$$

we can conclude that

$$(\Delta p)^2 = (\Delta mv)^2 + (\Delta mu)^2. \tag{24}$$

In particular,

$$\Delta p \geq \Delta mu. \tag{25}$$

Namely, the Heisenberg position-momentum uncertainty relations

$$[\langle\psi(t), x^2\psi(t)\rangle - \langle\psi(t), x\psi(t)\rangle^2]^{1/2} \times [\langle\psi(t), p^2\psi(t)\rangle - \langle\psi(t), p\psi(t)\rangle^2]^{1/2} \geq \hbar/2 \tag{26}$$

can be traced back to the position–osmotic velocity uncertainty relation

$$\{E(q^2(t)) - [E(q(t))]^2\}^{1/2}\left[E\left(\left(\frac{p_+(t)-p_-(t)}{2}\right)^2\right)\right]^{1/2} \geq \frac{\hbar}{2} \tag{27}$$

for the stochastic process $q(t)$ associated with the wave function $\psi(x,t)$ in the sense of Nelson.

As a final physical remark, we wish to observe that out of the two terms of the decomposition (24) of the root mean square deviation of the quantum-mechanical momentum there is a part due to the current velocity and a part due to the osmotic velocity; it is just the osmotic term that forces the position-momentum uncertainty.

[1]E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton Univ. Press, Princeton, 1967).
[2]*New Stochastic Methods in Physics*, edited by C. DeWitt-Morette and K. D. Elworthy, Phys. Rep. 77C, No. 3 (1981) (in particular, for a more detailed description of the concepts of stochastic mechanics, see the contribution of F. Guerra, p. 263).
[3]R. Fürth, Z. Phys. 81, 143–162 (1933).

Stochastic Method for the Numerical Study of Lattice Fermions

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A new stochastic method for the numerical study of lattice fermions is presented. Its efficiency is demonstrated on a field-theoretic model in four dimensions with coupled boson and fermion degrees of freedom. The exact fermion propagator is calculated and agrees very accurately with the numerical results of the stochastic procedure on finite lattices of 10^4 and $8^3 \times 16$ sites, respectively. The contribution of fermionic vacuum polarization to mass renormalization is evaluated with precision. The method is directly applicable to quantum chromodynamics.

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During the last twelve months we have witnessed considerable effort to develop Monte Carlo methods for the numerical study of quantum systems with fermionic degrees of freedom. This outstanding problem is of great importance for applications in quantum field theories, condensed matter physics, and nuclear physics.

Previous techniques[1-9] were slow when a very large number of fermionic degrees of freedom were involved, since the computational time re-

