Traversal Time for Tunneling

M. Büttiker and R. Landauer
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598
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One of several contradictory existing results for the time a tunneling particle interacts with its barrier is confirmed, by considering tunneling through a time-modulated barrier. At low modulation frequencies the traversing particle sees a static barrier. At high frequencies the particle tunnels through the time-averaged potential, but can do it inelastically, losing or gaining modulation quanta. The transition between the two regimes yields $\int dx\,[m/2(V-E)]^{1/2}$ for the traversal time.

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In 1932 MacColl[1] pointed out that tunneling is not only characterized by a tunneling rate, but also by a time during which the tunneling particle is actually traversing the barrier. There are many later contradictory results, and we can cite only a few.[2-11] The traversal time is of particular significance in many-body problems reduced to approximate single-particle problems, where we ask about the ability of some degrees of freedom to adjust to the progress of the tunneling. One of these problems, tunneling in the presence of friction, is of interest in Josephson junction circuits,[12,13] and will be discussed later. Another is the tunneling of an electron out of a metal, through an insulator, and the ability of the image charge on the metal surface to spread out as the electron departs from the surface.[9,14] The effective image force also depends on the extent to which contributions to the dielectric constant of the insulator can respond within the traversal time.[9]

We will substantiate *one* of several existing expressions for traversal time by studying the tunneling through a time-dependent barrier.

$$V(x,t)=V_0(x)+V_1(x)\cos\omega t. \tag{1}$$

$V_0(x)$ is static and $V_1(x)$ is the amplitude of a small modulation. Particles incident with energy E, interacting with the perturbation $V_1(x)\cos\omega t$, will emit or absorb modulation quanta $\hbar\omega$. Our key point is as follows: If the period in Eq. (1) is long compared to the time during which the particle interacts with the barrier, then the particle sees an effectively static barrier during its traversal. At frequencies high compared to the reciprocal traversal time the particle sees many cycles of the oscillation. Thus, the effective barrier is neither larger nor smaller, and the intensities of the transmitted beams differ only because the particles can absorb or give up modulation quanta. Particles gaining energy tunnel more easily through the barrier. As the modulation frequency is varied the crossover between the two types of behavior occurs when $\omega\tau\cong 1$, where τ is the interaction time of a *transmitted* particle. Thus we find an interaction time

$$\tau=\int_{x_1}^{x_2}[m/\hbar\kappa(x)]\,dx$$

$$=\int_{x_1}^{x_2}\{m/2[V_0(x)-E]\}^{1/2}dx. \tag{2}$$

x_1 and x_2 are turning points, m the mass, and $\kappa(x)=\{2m[V_0(x)-E]\}^{1/2}/\hbar$. Equation (2) is valid in the WKB limit at energies below the barrier peak. Well above the barrier we obtain $\tau=\int\{m/2[E-V_0(x)]\}^{1/2}dx$. This Letter emphasizes the limit where there is only a small probability for tunneling.

We restrict most of our discussion to a rectangular barrier with height V_0, and width d, centered at $x=0$. With an incident wave function of unit amplitude and a current $j=\hbar k/m$, where $k=(2mE)^{1/2}/\hbar$, the transmission probability for a static barrier is $T=[1+(k_0^4/4k^2\kappa^2)\sinh^2\kappa d]^{-1}$. Here $k_0=(2mV_0)^{1/2}/\hbar$ and $\kappa=(k_0^2-k^2)^{1/2}$ is the rate of exponential decay. For an almost completely reflecting barrier, hereafter called an opaque barrier, $\kappa d\gg 1$, and the transmission probability, for $E<V_0$, is given by

$$T=(16k^2\kappa^2/k_0^4)e^{-2\kappa d}. \tag{3}$$

We allude to three different earlier methods, citing only a few key papers for each. A time delay for a scattering process can be calculated by following the peak of a wave packet, via the method of stationary phase.[2] If $\Delta\varphi$ is the change of the phase across the barrier, the time found is $\tau_\varphi=\hbar\partial\Delta\varphi/\partial E$. For an opaque rectangular barrier, Hartman[3] uses this to find $\tau_\varphi=2m/\hbar k\kappa$. In contrast to Eq. (2), this result is independent of the thickness, and diverges as the incident kinetic energy goes to zero. The more general form of τ_φ, for a slowly varying potential, is also independent of the detailed shape of the barrier. There seems little physical justification, however, for the identification of incident peaks with transmitted peaks, particularly in the presence of the strong deformation[9,15] of a wave packet transmitted through a barrier.

An approach given by Smith,[4] and advanced by others,[5] yields a time τ_s as the ratio of the number of particles under the barrier to the incident flux. For an opaque rectangular barrier $\tau_s \cong \hbar k/V_0\kappa$. This tends to zero with the square root of the kinetic energy of the incident particles, and is independent of the thickness of the barrier. This method does not distinguish between particles which at the end of their stay in the forbidden region have been reflected, and those that were transmitted. This time is the average dwell time of a particle in the barrier, and is not the traversal time, if most particles are reflected. A third method[6,7] uses the Larmor precession as a clock to measure traversal time. A small magnetic field is confined to the barrier. Comparing the spin orientation of transmitted particles with the orientation of the incident beam, Rybachenko[7] finds the same result for an opaque rectangular barrier as Smith. One of us has shown that the notion of a Larmor precession during tunneling is not correct; a refined approach again yields the result of Eq. (2).[16] The many other earlier results include the statement that the traversal takes no time.

Results similar to Eq. (2) are anticipated in Refs. 8–11. Sokolow, Loskutow, and Ternow[8] base this on the expectation value of momentum obtained from the exponentially decaying wave function, within the barrier. Schnupp[9] criticizes this, but confirms it by numerical computation. Leggett[10] discusses Eq. (2) only for the case where E is at the minimum of the initial potential well, below the ground state of that well, and correctly points to the physical significance of this "bounce" time, which elsewhere appears only as a mathematical device. Unfortunately, at the particular value of the energy in Ref. 10, the integral in Eq. (2) diverges. Schulman[11] generalizes this formal "bounce" time expression to arbitrary energy, and arrives at Eq. (2). In contrast to the cited work, we consider an explicitly time-dependent Hamiltonian, with the potential of Eq. (1). In this part of our discussion $V_0(x)$ is a rectangular barrier, with $V_0(x)=V_0$, if $|x| \leq d/2$, and zero otherwise. The perturbation amplitude $V_1(x)$ is constant over the barrier, and zero elsewhere. A wave of unit amplitude is incident from the left.

We proceed by first considering the simpler Hamiltonian, $H=p^2/2m+V_0+V_1\cos\omega t$, *spatially uniform* along the whole x axis. If the time-independent problem $H_0=p^2/2m+V_0$ has $H_0\varphi_E=E\varphi_E$, then in the time-dependent case

$$\psi_\pm(x,t;E) = \varphi_E(x)\exp\left(-\frac{iEt}{\hbar}\right)\exp\left(-\frac{iV_1}{\hbar\omega}\sin\omega t\right). \tag{4}$$

Equation (4) applies *within* the barrier. There we have $E<V_0$, and $\varphi_E(x)=e^{\pm\kappa x}$. We can separate[17] $\psi(x,t;E)$ into components with energies $E\pm n\hbar\omega$,

$$\psi_\pm(x,t;E)=e^{\pm\kappa x}e^{-iEt/\hbar}\left[\sum_{n=-\infty}^{n=+\infty} J_n\left(\frac{V_1}{\hbar\omega}\right)e^{-in\omega t}\right]. \tag{5}$$

J_n is a Bessel function. The time modulation[18] of the potential gives rise to "sidebands" describing particles which have absorbed ($n>0$) or emitted ($n<0$) modulation quanta. We take V_1 as a perturbation, and only include first-order corrections to the time-independent case. For small $V_1/\hbar\omega$, Bessel functions behave as $J_n \propto (V_1/\hbar\omega)^{|n|}$. Equation (5) shows that the order in $V_1/\hbar\omega$ corresponds to the order of the sidebands. To first order we obtain, therefore, the first two sidebands at $E\pm\hbar\omega$. To find the solution for the oscillating rectangular barrier we match a superposition of incident and reflected waves, and also transmitted waves, at the three energies E, $E\pm\hbar\omega$, to solutions within the barrier. Within the barrier we have a superposition of two solutions of Eq. (5), corresponding to each of the three unperturbed energies E, $E\pm\hbar\omega$. For the transmitted wave, at the frequency $E/\hbar$, we recover the results of the static barrier, with a coefficient multiplying $e^{ikx-iEt/\hbar}$ given by

$$D=(4k\kappa/k_0^2)e^{-\kappa d}\exp\{-i\arctan[(\kappa^2-k^2)/2k\kappa]\}e^{-ikd}. \tag{6a}$$

For the transmitted waves at the frequencies $(E/h)\pm\omega$ we find, for the coefficients multiplying $\exp[ik_\pm x - i(E\pm\hbar\omega)t/\hbar]$,

$$D_\pm = \mp D(V_1/2\hbar\omega)e^{\mp i(m\omega d/2\hbar k)}(e^{\pm\omega\tau}-1). \tag{6b}$$

$\tau=(m/\hbar\kappa)d$ is the time it would take a particle with the *real* velocity $v=\hbar\kappa/m$ to traverse the barrier, and this is the time Eq. (2) yields for a rectangular barrier. Its role as a transition boundary between regimes is suggested by the final right-hand side factor in Eq. (6b), but will be discussed in more detail. To obtain Eq. (6b) we have, in addition, assumed that $\hbar\omega \ll E$, so that the wave vectors of the sidebands $k_\pm = [2m(E \pm \hbar\omega)]^{1/2}/\hbar \cong k \pm m\omega/\hbar k$, and assumed $\hbar\omega \ll V_0 - E$, so that $\kappa_\pm = \kappa \mp m\omega/\hbar\kappa$. The probability of transmission at the sideband energies determined from Eq. (6b) is

$$T_\pm = |D_\pm|^2 = (V_1/2\hbar\omega)^2(e^{\pm\omega\tau} - 1)^2 T, \tag{7}$$

with T given by Eq. (3). For frequencies small compared to the reciprocal traversal time, a particle, during its interaction with the barrier, sees only a static barrier. The wave function in the barrier decays with the instantaneous exponential WKB rate $\kappa(t) = \{2m[V(t) - E]\}^{1/2}/\hbar$. For small V_1 this gives $\kappa(t) = \kappa - (mV_1/\hbar^2\kappa)\cos\omega t$, where κ is the static decay constant. The instantaneous transmission coefficient is found by replacing κ in Eq. (6a) by $\kappa(t)$. The intensity for the resulting first two sidebands is then easily shown to be

$$T_\pm = (V_1\tau/2\hbar)^2 T, \tag{8}$$

and is obviously the low-frequency limit of Eq. (7).

At high frequencies the particle will see a time-averaged barrier of effective height V_0. More precisely: The intensity of the transmitted beam will be dominated by the fact that a particle which absorbs a quantum $\hbar\omega$ and thus has energy $E + \hbar\omega$ traverses the barrier more easily than particles with energy E or $E - \hbar\omega$. A particle incident on the static barrier with an energy $E + \hbar\omega$ has a transmission probability $T_{E+\hbar\omega} = Te^{2\omega\tau}$, ignoring corrections of order $(\hbar\omega/E)^2$ and $(\hbar\omega/V_0 - E)^2$. Thus the intensity of the upper sideband, in Eq. (7), for high frequencies, is given by $T_+ = (V_1/2\hbar\omega)^2 T_{E+\hbar\omega}$. The appearance of $T_{E+\hbar\omega}$ may be puzzling, because it characterizes transmission through the *whole* barrier, at the higher energy, whereas the quanta $\hbar\omega$ can be absorbed anywhere along the barrier. In view, however, of the decreased exponential decay at the higher energy, it is clear that modulation quanta absorbed near the incident end will dominate the upper sideband. Similarly the dominant contribution to the lower sideband comes from particles that emit modulation quanta near $x = d/2$. Indeed $T_- = (V_1/2\hbar\omega)^2 T$, at high frequencies, depends on the length of the barrier only through T. The transition from Eq. (8), with equal values for the two transmission coefficients, to very unequal transmission probabilities at high frequencies is best exhibited by writing Eq. (7) in the form $(T_+ - T_-)/(T_+ + T_-) = \tanh\omega\tau$. Thus τ specifies the crossover.

The WKB approximation allows an extension to barriers of a more general shape. $\psi = Ae^{iS/\hbar}$ is determined by a solution of the time-dependent Hamilton-Jacobi equation

$$\partial S/\partial t = (2m)^{-1}(\partial S/\partial x)^2 - V. \tag{9}$$

Take $S = S_0 + s$, where $S_0 = -Et + i\hbar\int \kappa(x)dx$ is the solution for the static case, and s arises from the modulation. To first order in $V_1(x)$, confined to the barrier, we find

$$s(x) = i\left(\frac{m}{2}\right)\left[e^{i\omega t}\int_{x_0}^{x} dx' \frac{V_1(x')}{\hbar\kappa(x')} \exp\left(-\int_{x'}^{x} dx'' \frac{m\omega}{\hbar\kappa(x'')}\right) + (\omega \to -\omega)\right], \tag{10}$$

where $(\omega \to -\omega)$ denotes an expression obtained by replacing ω by $-\omega$ in the preceding term. x_0 is the left-end turning point of the barrier, where we take $s = 0$. For small $V_1(x)$ we are again left with only two sidebands. The exponential factor in the first right-hand-side term of Eq. (10) describes the increased attenuation of the lower sideband term, and has the form $\exp[-\int_{x'}^{x} dx''\,\omega/v]$, with $v = \hbar\kappa(x)/m$. Thus, as in Eq. (7), the velocity obtained from the wave-function decay rate, combined with the barrier thickness, determines the relevant transition time.

For dissipative tunneling, mentioned earlier, we use the traversal time to obtain an estimate of the effects of friction on transmission. The energy loss of a particle with velocity v and friction coefficient γ is given by $\Delta E = \gamma\int_0^x v(x')dx'$. A particle tunneling through a barrier does not, of course, have a well defined velocity. Our traversal time, however, defines an effective velocity $v = \hbar\kappa(x)/m$. For small dissipation we can evaluate ΔE with the help of the velocity $v_0(x)$ for the undamped system. Thus, the tunneling particle loses energy $\Delta E(x) = \gamma\int_0^x [\hbar\kappa_0(x')/m]dx'$ as the particle traverses the barrier. To find the transmission probability we assume that the effective WKB decay rate κ, in the presence of damping, is still given by $\kappa = \{2m[V(x) - E(x)]\}^{1/2}/\hbar$ where $E(x) = E_0 - \Delta E(x)$ is the energy corrected for damping. The use of the WKB phase integral in the presence of diminishing energy is an *ad hoc* procedure requiring more justification than we provide. It is, however, made plausible by the very similar behavior for a particle which gives up a modulation quantum. To first order in γ we find $\hbar\kappa = \hbar\kappa_0 + [\gamma/v_0(x)]\int_0^x v_0(x')dx'$. This gives a phase integral for the exponential attenuation of the wave function,

$$S = \int \hbar\kappa(x)dx$$
$$= S_0 + \gamma\int_0^x [dx'/v_0(x')]\int_0^{x'} v_0(x'')dx''. \qquad (11)$$

If $v_0(x)$ is smooth and simple then the dissipative effects provide a contribution in Eq. (11) of the order of γd^2, where d is the tunneling distance. Thus dissipation causes decreased transmission. A decrease of this order has been predicted by Caldeira and Leggett.[12]

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