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Entropy Evaporated by a Black Hole

W. H. Zurek
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125
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It is shown that the entropy of the radiation evaporated by an uncharged, nonrotating black hole into vacuum in the course of its lifetime is approximately $\frac{4}{3}$ times the initial entropy of this black hole. Also considered is a thermodynamically reversible process in which an increase of black-hole entropy is equal to the decrease of the entropy of its surroundings. Implications of these results for the generalized second law of thermodynamics and for the interpretation of black-hole entropy are pointed out.

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Bekenstein has conjectured that the entropy of a black hole can be regarded as a measure of ignorance of an observer located outside of its horizon about the black hole's internal state, or, alternatively, about the initial, precollapse phase-space configuration which has led to the formation of this black hole[1]: The ignorance of an external observer is assured by the "black hole has no hair" theorem,[2] which implies that no observer remaining outside a black hole can know more than its mass, angular momentum, and its charge. However, Bekenstein reasoned, a specific black hole must have originated from some specific object. Moreover, an observer curious enough to cross the horizon and become, irreversibly, an "internal observer," could, presumably, learn about the black hole's internal state. This internal state should be related to the precollapse state. Therefore, Bekenstein's conjecture can be backed up with the following two plausible assertions: (1) The entropy of the black hole, S_{BH}, is related to the number of the internal configurations of the black hole, W, by the Boltzmann formula, $S_{BH} = \ln W$. (2) There is a one-to-one correspondence between the internal state and the precollapse configuration.

So far, it has proved difficult to turn this intuitive argument into a rigorous derivation of black-hole entropy. The crux of this difficulty lies in devising a reliable count of "the number of initial configurations that result in a formation of the same black hole." A count of the "internal states of black hole" is, at present, equally elusive.[3] Thus, so far, no one has been able to confirm the expression for the entropy of the black hole obtained by Hawking,[4] $S_{BH} = A/4$, where A is the area of the black-hole horizon, with a rigorous and independent estimate of W. The aim of this paper is to present two calculations directly relevant to Bekenstein's conjecture: (1) I shall estimate the total number of configurations evaporated by the black hole into vacuum, and I shall find that their entropy exceeds the entropy lost by the black hole by a factor $\sim\frac{4}{3}$; (2) I shall show that in a thermodynamically reversible process the total number of configurations absorbed by a black hole can be matched as closely as desired by the increase of the black-hole entropy. This, in effect, proves Bekenstein's conjecture. Neither of our arguments could have been made by Bekenstein at the time when he put forward his conjecture[1,5]: Both of them are based on the phenomenon of black-hole radiation, which was proposed by Hawking some time later.[4]

To calculate the entropy of the radiation emitted by the black hole we start from Hawking's formula for the black-hole temperature[4]: $T_{\rm BH} = \kappa/2\pi$. Here κ stands for the surface gravity of the black hole. For an uncharged, nonrotating black hole, which will be considered below, entropy and temperature are given directly in terms of the black-hole mass:

$$S_{\rm BH} = 4\pi M^2, \tag{1}$$

$$T_{\rm BH} = (8\pi M)^{-1}. \tag{2}$$

Black holes emit blackbody radiation at the temperature $T_{\rm BH}$.[4,6] Therefore, the radiation emitted in a short time interval dt carries an energy equal to $dE = \Sigma\alpha(T_{\rm BH})^4 dt$, and an entropy of $dS \approx \frac{4}{3}\Sigma\alpha(T_{\rm BH})^3 dt$. Here Σ is an effective cross section of a black hole and α is the usual Stefan constant. To compare the decrease of the black-hole entropy with the increase of the entropy of its surroundings I calculate

$$dS_{\rm BH} = 4\pi[M^2 - (M - dE)^2] = 8\pi M dE.$$

Now it is not difficult to evaluate the entropy production ratio R:

$$R = dS/dS_{\rm BH} \approx \tfrac{4}{3}. \tag{3}$$

This is the first key result of this paper. I shall confirm it below with a more precise estimate.

To arrive at a general formula for the entropy production ratio R applicable also in a more realistic situation when the emitted radiation is not exactly black body, we shall use thermodynamic identity $dE_{\rm BH} = T_{\rm BH}\, dS_{\rm BH}$. From this identity it follows that, when the radiation emitted during dt carries energy $dE = dE_{\rm BH}$, and some arbitrary amount of entropy, dS, the entropy production ratio must be

$$R = dS/dS_{\rm BH} = T_{\rm BH}\, dS/dE. \tag{4}$$

For blackbody radiation the right-hand side of Eq. (4) evaluates readily to $\frac{4}{3}$.

In case of the actual black-hole emission the most significant correction will result from the mode-dependent opacity of the black hole $\Gamma_{s,\omega,l,m,p}$, which, for a given black hole, is a function of the particle species s, energy ω, angular momentum (l, m) and helicity p. If we assume that the opacities are given, R can be computed once dS and dE are evaluated. According to Planck, the average entropy of a single bosonic mode is given by

$$S_{s,\omega,l,m,p} = (\langle n\rangle + 1)\ln(\langle n\rangle + 1) - \langle n\rangle \ln\langle n\rangle, \tag{5}$$

where $\langle n\rangle$ is the average number of quanta per mode.[7] For a black hole, $\langle n\rangle = \Gamma_{s,\omega,l,m,p}[\exp(\omega/T_{\rm BH}) - 1]^{-1}$.[4,6] The total entropy of the emitted radiation is therefore $dS \sim \sum_{s,\omega,l,m,p} S_{s,\omega,l,m,p}$. The corresponding energy is given by

$$dE \sim \sum_{s,\omega,l,m,p} \omega \Gamma_{s,\omega,l,m,p}[\exp(\omega/T_{\rm BH}) - 1]^{-1}.$$

The ratio dS/dE which plays such a key role in Eq. (4) is now given directly in terms of the temperature and of the mode opacities.

Calculation of the black-hole opacities is rather complicated and analytic expressions describing their behavior can be given only in the limits $\omega M \gg 1$ and $\omega M \ll 1$.[8] We can, nevertheless, assess the sensitivity of the entropy production ratio R to the mode-dependent opacity by retaining the key feature of the black-hole spectrum which distinguishes it from the genuine black body, that is, the relative deficiency of low-frequency quanta. To facilitate this calculation I write the rates of entropy and energy emission in an approximate form:

$$dS_{s,p}/dt = 27\pi M^2 \int_0^\infty d\omega\, \omega^2 \sigma_{s,p}(\omega, M) S_{s,\omega,p};$$

$$dE_{s,p}/dt = 27\pi M^2 \int_0^\infty d\omega\, \omega^3 \sigma_{s,p}(\omega, M)\langle n\rangle.$$

Above, $\sigma_{s,p}(\omega, M)$ is the absorptivity of a unit area of the optical section of the black hole, which is equal to $27\pi M^2$. It is defined in terms of the frequency-dependent total cross section of the black hole, $\Sigma_{s,p}(\omega, M)$, which can be expressed directly in terms of the mode opacities:

$$\sigma_{s,p}(\omega, M) = \Sigma_{s,p}(\omega, M)/(27\pi M^2) = \sum_{l,m} \Gamma_{s,\omega,l,m,p}/[27(\omega M)^2].$$

For massless fields $\sigma_{s,p}(\omega, M)$ is a function of ω and M only through the product ωM.[8] This important scaling property allows us to write the entropy production as a ratio of two integrals:

$$R = dS/dS_{\rm BH} = I/J, \tag{6}$$

where I and J are given by

$$I = \int_0^\infty dx\, x^2 \sigma(x) \frac{xe^x - (e^x - 1)\ln(e^x - 1)}{e^x - 1}; \tag{7}$$

$$J = \int_0^\infty dx\, x^3 \sigma(x)/(e^x - 1). \tag{8}$$

In the above $x = \omega/T_{\rm BH}$. For the simplest case, $\sigma(x) = 1$, both integrals can be evaluated analytically and their ratio is equal to $\frac{4}{3}$, in agreement with Eq. (3). One can also show that for an arbitrary nonnegative $\sigma(x)$ the ratio of I and J will be always greater than unity; the second law of thermodynamics, which requires $I/J>1$ if the black-hole evaporation is to proceed, is satisfied.

For massless fields, $\sigma(x)$, typically, approaches 0 in the limit of low-frequency quanta and has a high-frequency limit of 1.[8] A "first approximation" of this effect can be achieved by introducing a cutoff at some $x=X$. That is, $\sigma(x)=0$ for $x<X$ and otherwise $\sigma(x)=1$. The ratio R is now a function of this low-frequency cutoff, and can be readily obtained through a numerical computation. For example, $X=1$ yields $dS/dS_{\mathrm{BH}}=1.31$, while $X=10$ results in $dS/dS_{\mathrm{BH}}=1.09$. In general, as the cutoff moves towards the high-frequency limit, the entropy production ratio R approaches unity. On the other hand, if one would set $X=0$, and introduce instead a high-frequency cutoff, R could become arbitrarily large.

In view of the increase of entropy one might be tempted to reason that, since the entropy of the black hole, interpreted via Boltzmann's formula, is a measure of the number of its internal states, then the mapping from the internal state of the black hole to the final state of the emitted radiation cannot be one to one, and, therefore, cannot be achieved by means of a unitary S matrix. Consequently, evaporation of the black hole into vacuum appears to be fundamentally irreversible, with pure states evolving into genuine density matrices: Quantum theory of black-hole evaporation would seem to demand existence of a "superoperator."[9] Moreover, at first sight $R>1$ appears to settle a controversy on whether black-hole formation and its subsequent evaporation are "genuinely" irreversible.[9-13] Unfortunately, these fundamental questions cannot be answered, in our opinion, with arguments of purely thermodynamic nature. Both black-hole collapse and evaporation occur far from equilibrium, and an irreversible increase of entropy in their course is likely to be of the same nature as an increase of entropy in an evolving, ordinary many-body system, with a dynamics which is fundamentally reversible. Nothing short of a fully quantum description of the process of evaporation is likely to resolve these problems. For, the present approach in which the fields are quantized, but the background metric is classical, may be "forcing" evolutions which are fundamentally reversible to appear irreversible by implicitly "tracing out" correlations of emitted quanta with the quantum state of the background gravitational field. And there are reasons to suspect that some unaccounted for correlations are indeed present among the quanta emitted by the black hole. As circumstantial evidence for them one can regard the predicted impressions of an observer who falls freely into the black hole. For him radiation should disappear altogether, at least close to the horizon.[14] Clearly, this is not a behavior of the run-of-the-mill blackbody heat bath. The state of the field along the free-fall trajectories must be very special, and this may be considered as an indication of such correlations.

Most important corrections to our estimate of R will result from the interactions between quanta, as well as from the existence of massive fields. They are nevertheless unlikely to be large, and are definitely not going to alter the key conclusion: The entropy S of the debris left after the time τ when the black hole of mass M has evaporated completely into vacuum,

$$S=\int_0^\tau\left(\frac{dS}{dt}\right)dt=\int_0^M\left(\frac{dS}{dt}\right)\left(\frac{dE}{dt}\right)^{-1}dE\approx\tfrac{4}{3}S_{\mathrm{BH}}, \tag{9}$$

is $\sim 30\%$ larger than the entropy of the black hole.

When a black hole is placed in a "heat bath" of temperature Θ, changes of the energy and entropy of the black-hole surroundings caused by the radiation it emits and absorbs proceed approximately at the rate $dE/dt\sim a(T_{\mathrm{BH}}^4-\Theta^4)$, $dS/dt\sim\frac{4}{3}a(T_{\mathrm{BH}}^3-\Theta^3)$. The ratio between the entropy of the radiation evaporated by the black hole and its entropy loss is now given by the formula

$$R'(\Theta,T_{\mathrm{BH}})=dS/dS_{\mathrm{BH}}=dS/(8\pi M dE)$$
$$=\tfrac{4}{3}T_{\mathrm{BH}}(T_{\mathrm{BH}}^3-\Theta^3)/(T_{\mathrm{BH}}^4-\Theta^4). \tag{10}$$

In the vacuum limit, i.e., when $\Theta/T_{\mathrm{BH}}\to 0$, Eq. (10) goes over to the previously obtained Eq. (3). At the "other end," i.e., in the limit $\Theta/T_{\mathrm{BH}}\to\infty$ the ratio $dS/dS_{\mathrm{BH}}=\frac{4}{3}(T_{\mathrm{BH}}/\Theta)\to 0$.

The most interesting case occurs when $T_{\mathrm{BH}}\approx\Theta$. Then the black hole is nearly in equilibrium with its environment. For example, when $\Theta=T_{\mathrm{BH}}-\Delta$, the ratio of entropies becomes

$$dS/dS_{\mathrm{BH}}\approx 1+\Delta/2T_{\mathrm{BH}}. \tag{11}$$

Thus, the entropy production ratio can be made as small as one wishes by allowing the black hole to evaporate slowly into a heat bath of a temperature only slightly below the temperature of the black hole. These conclusions are in accord with the generalized second law of thermodynamics. They lead us to the second key result of our considerations: When the entropy is interpreted in terms of Boltzmann's formula, one is forced to conclude that in a thermodynamically reversible process disappearance of a certain number of configurations absorbed by the black hole from its environment results in the gain of the same number of internal configurations of the black hole, reflected in the increase of its entropy. This is precisely the thesis of the Bekenstein conjecture.

A "practical" realization of such thermodynamically reversible evolution of the black hole is easily devised: Consider a black hole insulated in a perfectly reflecting container. When the mass of the black hole, M, the volume of the container, V, and the total energy inside the container, U, satisfy the equation $U - M = aVT_{BH}{}^4 = aV(8\pi M)^{-4}$, the black hole is in equilibrium with the surrounding heat bath.[10, 15, 16] Such equilibrium will be indeed stable (a global extremum of entropy) when $M/U > 0.97702$. The size of the black hole can be now altered in an adiabatic fashion, by slowly changing the volume of the whole container. When this change is sufficiently slow, so that all the modes within the container remain close to equilibrium with the black hole, the increase of the total entropy of its contents will be negligible. This thermodynamic system can be used as a "black-hole engine": One of the walls of the container can be allowed to expand, much like a piston of an ordinary engine cylinder, under the pressure of the black-hole radiation. Unfortunately, with the current, scarce, supplies of Hawking-size black holes this process is unlikely to help in the solution of the energy crisis.

The aim of this paper was to show that the evaporation of the black hole will usually occur as an irreversible process, with an increase of the total entropy. This increase can be characterized by the entropy production ratio R, which for the black hole evaporating into vacuum is typically of the order of 1.3. However, black-hole evaporation can be also made thermodynamically reversible, with the entropy of the evaporated radiation equal to the entropy lost by the black hole. This last result confirms Bekenstein's conjecture, according to which the entropy of the black hole is a measure of the number of the precollapse configurations: It provides a direct proof of the inverse of this conjecture by showing that $\exp(S_{BH})$ configurations will be produced in a reversible evaporation of the black hole. Moreover, a reverse of such an evaporation process can be used to form a black hole. Therefore, the arguments verify as well the original conjecture of Bekenstein.

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