Intermittent Chaos in Josephson Junctions

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The intermittent-type chaos occurring in rf- and dc-current-driven Josephson junctions is investigated. A simple physical model is proposed and is used for an analytic calculation of the power spectrum. The latter has a broadband part which decays algebraically at high frequencies. Comparison with numerical results shows good agreement. Further application and generalizations of this approach are outlined.

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One of the most intriguing experimental features exhibited by some externally driven dissipative dynamical systems is the occurrence of sequences of periodic states separated by gaps of chaotic, intermittent nature. Such an effect has been recently observed in chemical reactions[1,2]; it appears in Josephson junctions[3,4] and may exist in Rayleigh-Bénard[5] and Taylor[1,5] systems. Motivated by the recently introduced ideas of universality in chaotic systems,[6] we analyze a *simple* model giving rise to such phenomena. We use the resistively shunted junction (RSJ) model[7] for the Josephson junction driven by both ac and dc current sources (or a pendulum with dissipation driven by external constant and periodic forces). The model is described by the following equation of motion[4,8]:

$$\ddot{\theta} + G\dot{\theta} + \sin\theta = I + A\sin(\omega_{ex}t), \tag{1}$$

where $G = (\omega_J RC)^{-1}$; $\omega_J^2 = 2eI_J/\hbar c$; θ is the phase difference across the junction; and R, C, and I_J are resistance, capacitance, and critical current of the junction, respectively. Time (t) is measured in units of ω_J^{-1}, and ω_{ex} is the external frequency measured in units of ω_J. I and A are measured in units of I_J.

Many aspects of this model have been investigated previously.[3,4,8-10] Here we shall focus on the case where the I-V characteristic is as shown in the inset in Fig. 1 (from Ref. 4). When the current I satisfies $I < I_0$ (step "0") the solution of Eq. (1) is periodic (a limit cycle) but with a vanishing $\langle \dot{\theta} \rangle$ (or average dc voltage $\bar{V}$). A different periodic $\bar{V} \neq 0$ solution holds for $I_1 < I < I_2$ (step "1"). When $I_0 < I < I_1$ (the lowest dotted region in Fig. 1), no periodic solution is stable and the system behaves "chaotically." Similar behavior occurs in higher gaps (dotted regions in the inset in Fig. 1). Figure 1 shows a plot of $\dot{\theta}$ vs time for $I_0 < I < I_1$ obtained from a numerical solution of Eq. (1). The plot shows a random appearance of "upper" and lower peaks, each peak occurring for a time roughly equal to the forcing period. The upper peaks resemble the type of oscillations occurring in step 0 whereas the lower peaks resemble the oscillations occurring in step 1. This leads us to propose the following physical picture: When we set the current inside the gap to $I_0 < I < I_1$, the solution hops between the two neighboring limit cycles corresponding to $I < I_0$ and $I_2 < I < I_1$. The solution makes several revolutions around one (unstable) cycle, then moves to the second one and so on. We now simplify the picture by assuming that each such revo-

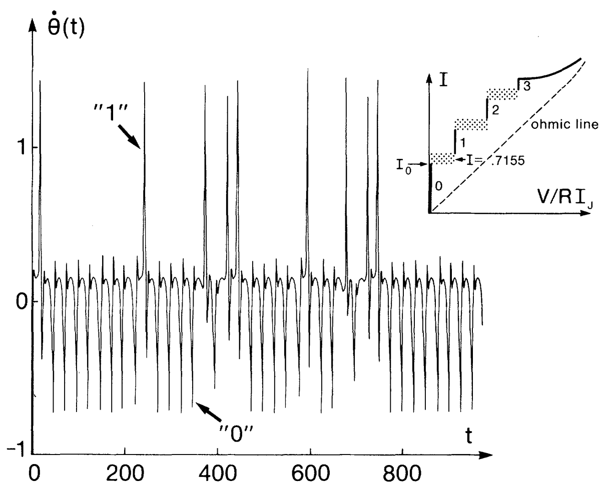

FIG. 1. Results of a numerical simulation of Eq. (1). The parameters are $G = 0.7$, $\omega_{ex} = 0.25$, $A = 0.4$, and $I = 0.7155$. Inset: a full I-V characteristic. The dotted regions in the inset are the chaotic gaps.

lution takes exactly a time T equal to the period of the external forcing time. We also *assume* that $\dot{\theta}=y_0(t)$ whenever the system is in a state corresponding to the step-0 cycle and likewise $\dot{\theta}=y_1(t)$ when it is in a step-1 cycle. $y_0(t)$ and $y_1(t)$ are periodic functions of period T. Finally, we *assume* that after completing a step-0 cycle the solution has a probability $P_{0,1}$ to move instantaneously to the step-1 cycle (a probability $1-P_{0,1}$ to repeat the 0 cycle once again). $P_{1,0}$ is defined similarly. This defines our model. An experimental or numerical verification of its consequences will provide a check for our assumed statistical nature of the dynamics.

One such consequence is the power spectrum $S(\omega)$:

$$S(\omega)\equiv \lim_{\bar{t}\to\infty}(1/\bar{t})\int_0^{\bar{t}}dt\int_{-\infty}^{\infty}d\tau\, e^{i\omega\tau}\langle f(t)f(t+\tau)\rangle, \tag{2}$$

where $f\equiv\dot{\theta}$ and the average (denoted by angular brackets) is over the previously defined ensemble with probabilities $P_{1,0}$, etc. Let us divide the time line axis into segments of length T. By assumption, in any interval $(n-1)T\leq t<nT$, n being an integer, $f(t)$ equals either y_0 or $y_1(t)$. Obviously, we can assume without loss of generality that t in Eq. (1) satisfies $0\leq t<T$ or $t=xT$, $0\leq x<1$. The time averaging with respect to the initial time $(1/\bar{t})\int_0^{\bar{t}}dt$ can then be replaced by $\int_0^1 dx$. Consequently

$$S(\omega)=\int_0^1 dx\sum_{n=-\infty}^{\infty}\int_{(n-x)T}^{(n+1-x)T}dt\, e^{i\omega\tau}\langle f(xT)f(xT+\tau)\rangle. \tag{3}$$

For a given n, $nT\leq xT+\tau<(n+1)T$. Thus for given n, $xT+\tau$ is in the nth segment. xT is always in the first segment. This means that $f(xT)$ can be replaced by $y_i(xT)$ and $f(xT+\tau)=y_j(xT+\tau)$, i,j being 0 or 1. Let us define $P(i,j,n)$ as the probability of the solution to be in the state (cycle) "i" in the first segment ($n=0$) *and* in the state "j" in the nth segment. The average in Eq. (2) can be replaced by

$$S(\omega)=\sum_{i,j=0}^{1}\int_0^1 dx\sum_{n=-\infty}^{\infty}\int_{(n-x)T}^{(n+1-x)T}d\tau\, e^{i\omega\tau}P(i,j,|n|)y_i(xT)y_j(xT+\tau). \tag{4}$$

Shifting the integration variable τ in the integrals in Eq. (4) by $(n-x)T$ and employing the assumed periodicity of y_i we obtain

$$S(\omega)=\sum_{i,j=0}^{1}\int_0^1 dx\sum_{n=-\infty}^{\infty}e^{i\omega(n-x)T}P(i,j,|n|)y_i(xT)\int_0^T e^{i\omega\tau}y_j(\tau)\,d\tau. \tag{5}$$

Defining

$$f_i(\omega)=\int_0^T e^{i\omega\tau}y_i(\tau)\,d\tau$$

we obtain

$$S(\omega)=(1/T)\sum_{n=-\infty}^{\infty}\sum_{i,j=0}^{1}e^{i\omega nT}P(i,j,|n|)f_j(\omega)f_i{}^*(\omega). \tag{6}$$

Next, we calculate $P(i,j,n)$. Let us define a 2×2 (transfer) matrix $\underline{F}$ by its elements: $F_{00}=1-P_{01}$, $F_{01}=P_{01}$, $F_{10}=P_{10}$, and $F_{11}=1-P_{10}$. The (i,j) element $(j,i=0,1)$ of $\underline{F}^N$ is the probability of having a j-type cycle in time segment $n=N$, provided the cycle in time segment $n=0$ is of type i. The probability of having a cycle of type i is $P_i\equiv P_{j,i}/(P_{i,j}+P_{j,i})$. Hence, $P(i,j,n)$ is given by

$$P(i,j,n)=[P_{j,i}/(P_{i,j}+P_{j,i})](\underline{F}^n)_{i,j}. \tag{7}$$

With use of the definition of the matrix $\underline{F}$ and Eq. (7), $P(i,j,n)$ is readily calculated as a function of i,j,n, P_{10}, and P_{01} with the result

$$P(0,0,n)=P_0{}^2+P_0P_1(1-P_{1,0}-P_{0,1})^n,\quad P(1,1,n)=P_1{}^2+P_0P_1(1-P_{1,0}-P_{0,1})^n,$$
$$P(1,0,n)=P(0,1,n)=P_1P_0-P_1P_0(1-P_{1,0}-P_{0,1})^n. \tag{8}$$

Next, the value of $P(i,j,n)$ is substituted into Eq. (6), and the summation is performed. The result is

$$S(\omega)=(1/T)|P_0f_0(\omega)+P_1f_1(\omega)|^2\sum_{n=-\infty}^{\infty}\delta(\omega T-2\pi n)+(1/T)|f_0(\omega)-f_1(\omega)|^2P_0P_1Q(\omega), \tag{9}$$

where

$$Q(\omega) = [1 - (1 - P_{01} - P_{10})^2]/[1 + (1 - P_{01} - P_{10})^2 - 2\cos(\omega T)(1 - P_{01} - P_{10})] .$$

We note that $S(\omega)$ consists of two parts—a series of δ functions at the fundamental frequency and its harmonics, weighted by a "form factor" $|P_0 f_0 + P_1 f_1|^2$, and a "continuous" broadband part. The periodic-type part of the spectrum is due to the system spending integer numbers of cycles in the two "states." In reality these δ functions should be somewhat broadened, e.g., by dephasing while hopping between states and by amplitude modulations. The broadband part of the spectrum is, of course, a characteristic of the chaotic nature of the solution.

In order to understand the nature of a typical spectrum we now employ simple functional forms for $y_0(t)$ and $y_1(t)$. Since we know[4,11] that in the 0 state there is no phase change after the completion of a cycle whereas in the step-1 state the total phase change in a cycle is 2π, we assume

$$y_0(t) = a_0 \sin(2\pi t/T + \varphi_0) ,$$
$$y_1(t) = a_1 \sin(2\pi t/T + \varphi_1) + 2\pi/T . \qquad (10)$$

The *Ansatz* in Eq. (10) can be justified on the basis of a perturbation theory[11] (e.g., in a small A). Using it we can easily calculate the Fourier transforms $f_0(\omega)$ and $f_1(\omega)$ and substitute in Eq. (6). For high enough ω ($\omega \gg 2\pi/T$), $|f_0(\omega)|^2 \propto 1/\omega^2$ and $|f_1(\omega)|^2 \propto 1/\omega^4$. Thus the spectrum decays *algebraically* at high frequencies. If $a_1 \gg a_0$ the term $|f_0(\omega) - f_1(\omega)|^2$ will be dominated by $|f_1(\omega)|^2$ and thus the decay of the spectrum will go like $1/\omega^4$ for a broad range of frequencies. For asymptotically large frequencies the $1/\omega^2$ behavior dictated by the presence of $f_0(\omega)$ will dominate. In practice, for frequencies greater than the inverse hopping time between the two cycles (neglected in our model), $S(\omega)$ will decay in a stronger fashion. This type of algebraic decay of the power spectrum is reminiscent of the one actually observed in Bénard experiments.[5,12]

Finally, we present a comparison of our calculation with a numerical simulation of Eq. (1). The relevant parameters are given in the figure captions. Figure 2 shows a plot of $\ln S(\omega)$, obtained by numerically solving Eq. (1). The inset shows the $\langle \dot\theta(t)\dot\theta(0)\rangle$ correlations from which $S(\omega)$ is derived. Figure 3 shows the broadband part of $S(\omega)$ from Eq. (9) for two different sets of phases φ_0, φ_1 [see Eq. (10)]. Except for the oscillations in Fig. 3 that depend on the choice of phase the two figures agree very well. These oscillations are "smeared" in practice (Fig. 2) because the cycles do not always start at the same phase as assumed in the model.

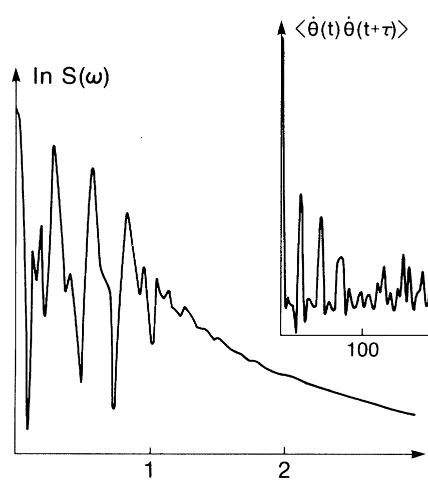

FIG. 2. $S(\omega)$ as calculated numerically from Eq. (1). The parameters are given in Fig. 1. Inset: the time correlation function $\langle \dot\theta(t), \dot\theta\rangle$, time being measured in units of $2\pi/\omega_J$.

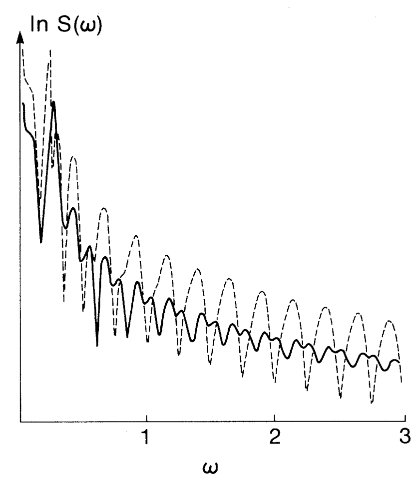

FIG. 3. The broadband part from Eq. (9). The parameters in Eq. (10) are $a_0 = 0.45$, $a_1 = 0.9$, $2\pi/T = 0.25$ for both curves and $\varphi_0 = 0$, $\varphi_1 = 1.5$ for the solid curve and $\varphi_0 = 0.25$, $\varphi_1 = 1$ for the broken curve. Note that the vertical units are the same in Figs. 2 and 3 but the zero of $\ln S(\omega)$ is shifted.

So far we have shown that the very simple idea of random hopping between two limit cycles can explain the general features of the spectrum of intermittent chaos in the Josephson junction. Noting that "hopping" between different portions of phase space is a very *general* phenomenon in many dynamical systems (e.g., Lorenz equations[13]), one of which exhibits $1/f$ noise,[14] it is interesting to ask how our simple model can be used as a starting point for modeling more aspects of intermittent chaos (e.g., the $1/f$ noise), in an analytic way. As noted by D'Humieres *et al.*,[15] *noise* plays a crucial role in modifying the results of our model. It is therefore interesting to include noise and dephasing (and perhaps hopping between more than two states) to obtain a better understanding of the phenomena underlying intermittent chaotic behavior. Parts of these aspects are under active investigation.

In conclusion, we have shown how a simple physical model can account for the power spectrum of an intermittent-type chaotic spectrum. The algebraic decay of the spectrum at high frequencies is easily understood on the basis of our model. Finally, we have mentioned possible generalizations of this work, which can help one to gain a better understanding (qualitatively as well as quantitatively) of the physics of intermittency.

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