Comment on "Magnetic Flux, Angular Momentum, and Statistics"

Wilczek[1] considers a particle of charge q in a circular orbit of radius ρ outside a line of magnetic flux Φ on the z axis. To show that the kinetic angular momentum l_z is quantized as $n - q\Phi/2\pi$, where n is an integer, he considers three methods. In the third method he makes a "singular gauge transformation" to eliminate the vector potential outside the z axis which imposes the new boundary condition $\psi'(\varphi + 2\pi) = \exp(-iq\Phi) \times\psi'(\varphi)$ on the wave function. It is this new boundary condition that gives the correct value for the kinetic angular momentum.

The purpose of this Comment is to draw the distinction between a change of representation of the operators that preserves the canonical commutation relations[2] and a gauge transformation.[3] In classical electrodynamics a gauge transformation on the potentials cannot change the electromagnetic field. In the problem considered by Wilczek the magnetic $\vec{B}$ is nonzero only on the z axis,

$$\vec{B} = \Phi\delta(x)\delta(y)\hat{z}. \tag{1}$$

A vector potential which gives this $\vec{B}$ is

$$\vec{A} = \hat{\varphi}\Phi/2\pi\rho, \tag{2}$$

where $\hat{\varphi}$ is a unit vector in the direction of the azimuthal angle φ. The singular gauge function used by Wilczek[1] is $\Lambda = \Phi\varphi/2\pi$ so that $\nabla\Lambda = \hat{\varphi}\Phi/2\pi\rho$. Therefore the new vector potential is $\vec{A}' = \vec{A} - \nabla\Lambda = 0$ everywhere, and the new magnetic field is $\vec{B}' = \nabla\times\vec{A}' = 0$ everywhere. The magnetic field has been changed, so $\Lambda = \Phi\varphi/2\pi$ is not a valid gauge function.[4-7]

On the other hand, the representation of the canonical momentum operator and wave function can be changed, which preserves the canonical commutation relations.[2,3] The Hilbert space for this problem is the set of quadratically integrable functions $L^2(S)$ on $S = \{(\rho,\varphi,z) | \rho = a > 0,\ \varphi \in [0,2\pi),\ z = 0\}$, where a is the radius of the orbit of the electron. The standard representation of the canonical angular momentum operator $p_\varphi = -i\partial/\partial\varphi$ can be changed by a unitary transformation to

$$p_\varphi' = \exp(-i\Gamma)p_\varphi \exp(i\Gamma) = p_\varphi + \partial\Gamma/\partial\varphi, \tag{3}$$

if the wave function is also changed to

$$\psi' = \exp(-i\Gamma)\psi, \tag{4}$$

where Γ is defined on S.

The kinetic angular momentum $l_z = p_\varphi - qA_\varphi$, where $A_\varphi = \rho\hat{\varphi}\cdot\vec{A}$, is the gauge-invariant operator corresponding to the observable angular momentum. If

$$\Gamma = q\Phi\varphi/2\pi \ \text{on}\ S, \tag{5}$$

then the new kinetic angular momentum operator on $L^2(S)$ is

$$l_z' = p_\varphi' - qA_\varphi = p_\varphi, \tag{6}$$

by Eqs. (2), (3), and (5). The new kinetic angular momentum is the old canonical angular momentum. The magnetic field and the vector potential are still given by Eqs. (1) and (2), respectively, so the electrodynamics has not been changed. The wave function ψ' satisfies the free-field Schrödinger equation, but with the boundary condition

$$\psi'(2\pi) = \exp(-iq\Phi)\psi'(0), \tag{7}$$

since $\psi'(0) = \psi(0)$ from Eqs. (4) and (5).

The eigenfunction of the new kinetic angular momentum operator in Eq. (6) which satisfies the boundary condition in Eq. (7) is

$$\psi_n'(\varphi) = (2\pi)^{-1/2} \exp[i(n - q\Phi/2\pi)\varphi], \tag{8}$$

where n is an integer. The eigenvalue of the kinetic angular momentum in Eq. (6) is thus $n - q\Phi/2\pi$, the same as obtained by Wilczek.[1]

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