Instabilities of an *m*-Vector Spin-Glass in a Field

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It is demonstrated that for a vector spin-glass in a magnetic field, replica-symmetry breaking, the theoretical indicator for magnetic irreversibility, occurs simultaneously with transverse spin-glass ordering as the temperature is reduced. It is argued that the irreversibility onset will be strong in the transverse direction, but only weak longitudinally, with a crossover to strong longitudinal irreversibility at a lower temperature.

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There is currently great experimental and theoretical interest in spin-glasses,[1] partly because randomness is an essential and not just a complicating ingredient in their recipe, but even more because they exhibit dramatic history dependence and apparent breakdown of ergodicity. As experimental evidence of the history dependence we note that the magnetization response to an applied field is very different if the field is applied before or after cooling from the paramagnetic state; the difference between the field-cooled and the quasi-instantaneous zero-field–cooled magnetization is referred to as the irreversible magnetization and grows continuously as the temperature is lowered past the spin-glass temperature. The time taken to reach the appropriate equilibrium (field-cooled) state grows with the size of the system, causing nonergodicity.

A common theoretical approach to spin-glasses is to map the physical disordered system into an effective pure one involving replicated spins interacting with one another through a more complicated interaction. Intuitively one expects a symmetry between replicas, but the breakdown of this symmetry is now recognized as an indicator for the history dependence mentioned above. The model which has been the basis for most analyses is an Ising one[2] in which the symmetry between replicas is broken[3] as soon as one enters the randomly frozen spin phase. Recently, however, Gabay and Toulouse (GT)[4] suggested that the corresponding vector-spin model in an external field should have two transitions as the temperature is lowered, first to a state with transverse spin-glass–like ordering but with symmetry between replicas and thus no history effects, followed at a lower temperature by a transition in which the replica symmetry is broken and irreversibility ensues. This suggestion has stimulated much experimental activity. In fact, as we show below, the symmetry between replicas is broken as soon as the transverse ordering occurs. Doubt is thus cast on the existence of a second transition, but we argue that there is likely to be a crossover in the longitudinal magnetic irreversibility from a weak onset at the transverse ordering transition to a strong form (analogous to the Ising case) by a temperature of the order of the lower one of GT. In contrast, strong transverse irreversibility is expected to commence immediately at the first transition. These observations are of importance to current experiments and suggest further tests of spin-glass replica modeling.

As did GT, we base our theory on the m-vector Sherrington-Kirkpatrick[2] (SK) model in a field,

$$\mathcal{H} = -\sum_{(ij)} J_{ij} \vec{S}_i \cdot \vec{S}_j - \sum_i \vec{H} \cdot \vec{S}_i , \qquad (1)$$

where the J_{ij} are quenched, independently random exchanges distributed with zero mean and variance $J/\sqrt{N}$. We choose units with $J = k_B = 1$, $|S|^2 = m$. Mean-field theory is believed to be exact for this model, and its solution yields the mean-field solution to a short-range model. We use the now-conventional replica analysis, treated within the replica-symmetric (RS) approximation with interreplica mode softening as the signal for irreversibility. Because of the deficiencies in the original GT treatment we provide sufficient detail to enable the reader to check its correctness.

Before making any RS assumptions, the free energy per spin is given by

$$f = -T \lim_{n\to 0} n^{-1} \max\{-\tfrac{1}{2}\beta^2 \sum_\mu [\sum_{(\alpha\beta)} (q_\mu^{(\alpha\beta)})^2 + \tfrac{1}{2}\sum_\alpha (q_\mu^{\alpha})^2] - \ln Q\}, \tag{2}$$

where

$$Q = \mathrm{Tr}_n \exp\{\beta^2 \sum_\mu [\sum_{(\alpha\beta)} q_\mu^{(\alpha\beta)} S_\mu^\alpha S_\mu^\beta + \tfrac{1}{2}\sum_\alpha q_\mu^\alpha (S_\mu^\alpha)^2] + \beta H \sum_\alpha S_1^\alpha]\}. \tag{3}$$

The subscripts μ, ν label Cartesian directions in m space, with the field direction being chosen as $\mu = 1$. α, β label replicas, and the notation $(\alpha\beta)$ refers to pairs of *different* labels. "max" signifies maximization with respect to the q's.

Within the replica-symmetric approximation we need only three parameters,[5]

$$q_\mu^\alpha = \lim_{n\to 0} \langle (S_\mu^\alpha)^2\rangle_n = 1 + (m\,\delta_{\mu,1} - 1)x, \quad q_\mu^{(\alpha\beta)} = \lim_{n\to 0} \langle S_\mu^\alpha S_\mu^\beta\rangle_n = q + (q_1 - q)\delta_{\mu,1}. \tag{4}$$

In the absence of a field the quadrupolar parameter x is zero, while q and q_1 are equal and become nonzero continuously as T is lowered past unity into an isotropic spin-glass phase. In a field both x and q_1 are nonzero for all T, but q has a phase transition at a field-dependent temperature, the higher one of GT.[6]

Within the RS approximation any disorder-averaged product of thermodynamic averages is simply related to an average in replica space[7]:

$$\langle \langle S_1^p \cdots S_m^q\rangle \cdots \langle S_1^r \cdots\rangle\rangle_J = \lim_{n\to 0} \langle (S_1^\alpha)^p \cdots (S_m^\alpha)^q \cdots (S_1^\beta)^r \cdots \rangle_n; \quad \alpha \neq \beta,$$

$$= \int_{-\infty}^{\infty} \cdots \int \prod_\mu \{[dt_\mu/(2\pi)^{1/2}] \exp(-t_\mu^2/2)\} (Z^{-1} \partial^{p+\cdots+q} Z/\partial a_1^p \cdots \partial a_n^p) \cdots (Z^{-1}\partial^{r+\cdots} Z/\partial a_1^r \cdots), \tag{5}$$

where

$$Z = \mathrm{Tr}\exp(\textstyle\sum_\mu a_\mu S_\mu + bS_1^2), \quad a_\mu = \beta[(q_\mu)^{1/2} t_\mu + H\delta_{\mu,1}], \quad b = \beta^2(q - q_1 + mx)/2. \tag{6}$$

$\langle\cdots\rangle$ denotes a thermodynamic average of the real system, $\langle\cdots\rangle_J$ a quenched average over the J distribution, and $\langle\cdots\rangle_n$ a thermal average in the replicated system.

Z reduces to a single integral,

$$Z = \sqrt{m}(2\pi)^{(m-1)/2} (|a|_{m-1})^{(3-m)/2} \int_{-\sqrt{m}}^{\sqrt{m}} dS \exp(a_1 S + bS^2)(m - S^2)^{(m-3)/4} I_{(m-3)/2}(|a|_{m-1}(m-S^2)^{1/2}), \tag{7}$$

where $|a|_{m-1} = (a_2^2 + \cdots + a_m^2)^{1/2}$ and $I_\nu(z)$ is a modified Bessel function of the first kind.

A particular application of (5) in (4) yields self-consistency equations for x, q, and q_1. The equation for the phase line on which the $q = 0$ transition occurs is given by

$$(m-1)^2/\beta^2 = \int_{-\infty}^{\infty} [dt_1/(2\pi)^{1/2}] \exp(-t_1^2/2)(P_{20}/P_{00})^2, \tag{8}$$

where

$$P_{np} = \int_{-\sqrt{m}}^{\sqrt{m}} dS \exp(a_1 S + b_0 S^2)(m - S^2)^{(m-3+n)/2} S^p, \quad b_0 = \beta^2(mx - q_1)/2,$$

and x and q_1 are determined via

$$1 + (m-1)x = \int_{-\infty}^{\infty} [dt_1/(2\pi)^{1/2}] \exp(-t_1^2/2)(m - P_{20}/P_{00}), \quad q_1 = \int_{-\infty}^{\infty} [dt_1/(2\pi)^{1/2}] \exp(-t_1^2/2)(P_{01}/P_{00})^2.$$

To lowest order in H the phase line for transverse freezing is[4,6]

$$T_{c1} = 1 - (m^2 + 4m + 2)H^2/4(m+2)^2; \tag{9}$$

the corresponding dominant behaviors of q_1 and x are

$$q_1 = |H|/\sqrt{2}, \quad x = H^2/4. \tag{10}$$

Let us now turn to the stability analysis. We expand the functions on the left-hand side of Eqs. (4) about their replica-symmetric extremal values and study the stability of the resulting free-energy functional to quadratic order in the deviations.[3] Explicitly, we take $x^\alpha = x + \epsilon^\alpha$, $q_\mu^{(\alpha\beta)} = q_\mu + \eta_\mu^{(\alpha\beta)}$, leading to the fluctuation Hamiltonian

$$L_2 = \tfrac{1}{2}\beta^2[\{\epsilon^\alpha\} \ \ \{\eta^{(\alpha\beta)}\}]\begin{bmatrix} A & B \\ B^T & C\end{bmatrix}\begin{bmatrix}\{\epsilon^\alpha\} \\ \{\eta^{(\alpha\beta)}\}\end{bmatrix}, \tag{11}$$

where

$$\begin{aligned} A^{\alpha\beta} &= \tfrac{1}{2}m(m-1)\delta_{\alpha\beta} - (m^2\beta^2/4)\{\lim_{n\to 0}\langle (S_1^\alpha)^2(S_1^\beta)^2\rangle_n - [1+(m-1)x]^2\}, \\ \beta_\mu^{\alpha(\beta\gamma)} &= -\tfrac{1}{2}m\beta^2\{\lim_{n\to 0}\langle (S_1^\alpha)^2 S_\mu^\beta S_\mu^\gamma\rangle_n - [1+(m-1)x]q_\mu\}, \\ C_{\mu\nu}^{(\alpha\beta)(\gamma\delta)} &= \delta_{(\alpha\beta)(\gamma\delta)}\delta_{\mu\nu} - \beta^2[\lim_{n\to 0}\langle S_\mu^\alpha S_\mu^\beta S_\nu^\gamma S_\nu^\delta\rangle - q_\mu q_\nu]. \end{aligned} \tag{12}$$

The eigenfunctions responsible for replica-symmetry breaking have all the ϵ^α zero and

$$\eta_\mu^{(\alpha\beta)} = \begin{cases} c_\mu; & (\alpha\beta) = (\theta_\mu \nu_\mu), \\ (2-n)^{-1}c_\mu; & \alpha \text{ or } \beta = \theta_\mu \text{ or } \nu_\mu \text{ but not both}, \\ 2(2-n)^{-1}(3-n)^{-1}c_\mu; & \alpha,\beta \neq \theta_\mu, \nu_\mu. \end{cases} \tag{13}$$

Their normal mode spectrum follows from the set of m equations

$$\textstyle\sum_\nu (\delta_{\mu\nu} - \chi_{\mu\nu}^{(2)})c_\nu = \lambda c_\mu, \tag{14}$$

where

$$\chi_{\mu\nu}^{(2)} = \beta^2\langle\langle S_\mu S_\nu\rangle - \langle S_\mu\rangle\langle S_\nu\rangle\rangle^2\rangle_J, \tag{15}$$

evaluated in the replica-symmetric approximation.

For $T > T_{c1}$ all the λ are positive and replica symmetry is stable. At $T = T_{c1}$ one mode becomes soft and for $T < T_{c1}$ has a negative eigenvalue, signaling instability. Specifically, for T just less than T_{c1} and H small,

$$\begin{aligned} &\chi_{11}^{(2)} = 1 - H^2/T^2 q_1 + 12 q_1^2/(m+2)^2T^6 + \ldots, \quad \chi_{\mu\mu}^{(2)} = 1 + 12q^2/(m+2)^2T^6 + \ldots; \quad \mu \neq 1, \\ &\chi_{1\mu}^{(2)} = 4q(H^2 + q_1)/(m+2)^2T^6 + \ldots; \quad \mu\neq 1, \quad \chi_{\mu\nu}^{(2)} = 4q^2/(m+2)^2T^6 + \ldots; \quad \mu\neq\nu\neq 1, \end{aligned} \tag{16}$$

so that to lowest order in q the lowest eigenvalue is

$$\lambda = -[4(m+1)/(m+2)^2 + 4\sqrt{2}|H|(m-1)/(m+2)^4 + O(H^2)]q^2 \tag{17}$$

and is negative. The corresponding eigenfunction is given by

$$(c_1/c_\mu) = 2q(m-1)/(m+2)^2; \quad \mu\neq 1. \tag{18}$$

As expected on physical grounds, the replica-symmetry breaking is symmetric in the hyperplane perpendicular to the field axis. Note also that, although (18) has its dominant components in $\mu\neq 1$, there is a component of symmetry breaking induced in the longitudinal direction for arbitrarily small q.

More generally, there are two eigenfunctions of (15) which are symmetric in the plane perpendicular to the field. Their eigenvalues are given by

$$(1-\chi_{11}^{(2)} - \lambda)[1 - \chi_{\mu\mu}^{(2)} - (m-2)\chi_{\mu\nu}^{(2)} - \lambda] - (m-1)(\chi_{1\mu}^{(2)})^2 = 0; \quad \mu\neq\nu\neq 1. \tag{19}$$

The lower of these eigenvalues is that discussed above. The other is essentially the mode considered by Gabay and Toulouse.[8] For small H, the temperature at which it becomes zero is given by

$$H^2 = 8(1-T_{c2})^3/(m+1)(m+2). \tag{20}$$

At this temperature the two eigenfunctions satisfy

$$(c_1/c_\mu)_L = (m-1)/(m+1), \quad (c_1/c_\mu)_U = -(m+1),$$

where U and L refer to upper and lower energy eigenfunctions. We see that both modes have significant longitudinal components at this temperature.

Let us now try to interpret our results. As soon as replica symmetry is broken, it becomes a difficult task to perform any analytical calculations, as one knows from the Ising case,[9] but there is now at least a folklore of implications on which we can draw. Replica-symmetry breaking appears to imply,[9] as its most direct physical manifestation, a difference between the equilibrium and the reversible (linear response) susceptibility. It would appear then that, below the transverse freezing transition temperature T_{c1}, the magnetization might not rotate bodily, but lag, when a small transverse field is applied (say, in an ac experiment, in contradistinction to an equilibrium measurement). In essence, it seems to predict the onset of a spontaneous effective dynamic anisotropy, with a restoring force preventing free rotation of the magnetization (in the absence of any anisotropic interaction in the Hamiltonian).

A second aspect concerns the question of the existence of a second transition at a lower temperature. If the transverse freezing line went continuously to the Ising instability line, when $m \to 1$, there would remain little argument in favor of a second transition. However, this is not the case. Rather T_{c2} of Eq. (20) goes over to the Ising line as $m \to 1$, so that continuity arguments alone would make it not inconceivable that some sort of crossover will remain. The character of the eigenfunctions suggests that such a crossover is highly probable. Further circumstantial evidence supporting the crossover concept comes from a study of a spin-glass with uniaxial anisotropy, but this will be reported separately.[10] The most likely physical effect associated with the crossover would be a marked increase in the difference between the two *longitudinal* susceptibilities (equilibrium and reversible).

Toulouse[11] has shown that the properties of a spin-glass with a mean ferromagnetic exchange can be deduced from those of a spin-glass with zero mean exchange but in a field. Such an analysis leads to the corollary that in an m-vector spin-glass system with mean ferromagnetic exchange there can occur a mixed ferromagnet–spin-glass phase.[4] Our observations lead to the conclusion that everywhere in such a phase replica symmetry will be broken, with its attendant history dependence, remanence, and slow relaxation behavior. Our speculations concerning crossover do, however, suggest that there is likely to be a crossover in the longitudinal irreversibility within the mixed-phase region, along a line analogous to that which separates the two mixed phases of Fig. 2 in Ref. 4.

In conclusion, once more, the mean-field theory for spin-glasses appears to be subtler than expected. For best use of its conclusions as guides to the study of real materials, there are two provisos: It is just a mean-field theory and, despite continuous progress, it remains one which is not completely resolved.

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