Relativistic Tidal Forces and the Possibility of Measuring Them

Bahram Mashhoon and Dietmar S. Theiss
Institut für Theoretische Physik, Universität zu Köln, D-5000 Cologne 41, Federal Republic of Germany
(Received 21 June 1982)

The relativistic corrections to the Newtonian tidal accelerations generated by a rotating system are studied. The possibility of testing the relativistic theory of gravitation by measuring such effects in a laboratory in orbit around the Earth is considered. A recent proposal to measure a rotation-dependent tidal acceleration as an alternative to the Stanford gyroscope experiment is critically examined and it is shown that such an experiment does not circumvent the basic difficulties associated with the gyroscope experiment.

PACS numbers: 04.20.Cv, 04.80.+z, 95.30.Sf

In the relativistic theory of gravitation, as well as in Newton's theory, a local system of coordinates ("laboratory") may be chosen which is inertial except for the presence of tidal forces. A general theory of tides has been developed based on an extension of the concept of a local Fermi frame.[1] In strong fields, the relativistic tidal effects are predicted to lead to interesting phenomena such as the emission of tidal gravitational radiation.[1] Relativistic corrections to the Newtonian tidal accelerations caused by a massive rotating source (such as the Earth) will be considered in this paper and the possibility of measuring these effects will be critically examined.

To interpret the results of measurements in a laboratory frame in terms of a local clock and locally determined spatial directions, it is necessary to refer the (covariant) equations of motion to a local tetrad frame consistent with the measurement procedure. The simplest possibility is to carry along a set of three (orthogonal) gyroscopes and to characterize all local events by a Fermi coordinate system. Let[2] $\lambda_{(\alpha)}{}^{\mu}$ be such a tetrad system so that $\lambda_{(0)}{}^{\mu}=\lambda^{\mu}$ is the tangent vector of a representative path (e.g., the center of mass) and $\lambda_{(i)}{}^{\mu}$ are the spatial ("gyroscope") directions. The scalar tidal accelerations take the form $-k_{ij}x^i$ under the conditions of interest here (see Ref. 1 for details). Here x^i are the local spatial coordinates and the symmetric tidal matrix k is given by

$$k_{ij}=R_{\mu\nu\rho\sigma}\lambda_{(i)}{}^{\mu}\lambda^{\nu}\lambda_{(j)}{}^{\rho}\lambda^{\sigma}. \qquad (1)$$

Thus the local "Newtonian" equations of motion (in terms of the proper time τ) should be supplemented by this tidal force. It usually proves convenient to use spatial axes $\Lambda_{(i)}{}^{\mu}=M_i{}^j(\tau)\lambda_{(j)}{}^{\mu}$, where M is an orthogonal matrix. The local equations of motion are now simply those in a rotating system with the tidal matrix given by

$$K=MkM^{\dagger}, \qquad (2)$$

The tidal matrix depends on the moments of the source; for a rotating mass distribution, one may separate the contributions of the mass and angular momentum. The latter effect may be attributed to a gravitational "magnetic" field which is anticipated on the basis of a certain analogy with electrodynamics.[3] The nature of such a field was first elucidated (to linear order in angular momentum) in the framework of Einstein's theory by Thirring and Lense.[4] Efforts aimed at obtaining observational evidence for this field have concentrated on an experiment to measure the cumulative effect of the precession of a gyroscope in orbit around the Earth with respect to fixed stars.[5] The frequency of precession consists of a geodetic term and a mass-current term caused by the mass (M) and angular momentum (J) of the source, and reflected in the metric perturbations,

$$\phi=GM/c^2r \quad\text{and}\quad \psi=GJ/c^3r^2, \qquad (3)$$

respectively. The geodetic ("Schwarzschild") term is essentially the Thomas precession for a "Keplerian" orbit, and hence the precession is due to a coupling between the gyroscope's spin and its orbit. The mass-current ("Lense-Thirring") term is due to the coupling of the gyroscope's spin and the angular momentum of the source. The two effects, of amplitudes $A_S=c\phi^{3/2}/r$ and $A_{LT}=c\psi/r$, are orthogonal for a polar orbit, parallel for an equatorial orbit, and a mixture of these for an inclined orbit. The basic requirements for testing the predictions of Einstein's theory for the two effects are that the extraneous torques on the gyroscope must be controlled such that the resulting precession amplitudes are well below A_S and A_{LT} levels, respectively.[6]

In a recent paper, Braginsky and Polnarev[7] have proposed an experiment to measure the influence of the rotation-dependent tidal acceleration on the oscillations of a spring in orbit around the Earth. According to these authors, such an experiment can circumvent the necessity of using "highly sensitive techniques" required for the gyroscope experiment. Our investigation shows, however, that the basic physical requirements for measurability are the same for the two experiments. To demonstrate this fact and to illustrate the nature of the effects involved, the tidal matrix will be given in this paper for an equatorial and a polar circular geodesic orbit in the field of a rotating mass approximated by the exterior Schwarzschild metric together with the Lense-Thirring term. The tidal matrix can then be written as

$$K = K^{\mathrm{N}}(1 + X^{\mathrm{S}} + X^{\mathrm{LT}}), \tag{4}$$

where the extra terms are the relativistic corrections to the Newtonian effect K^{N}. For a circular orbit with Keplerian frequency $\omega_0 = c\phi^{1/2}/r$, the Newtonian tidal matrix is $K^{\mathrm{N}} = \omega_0^2\sigma$, where σ is diagonal with respect to spatial axes that coincide with coordinate directions and $\sigma_{rr} = -2$, $\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = 1$.

Consider first an equatorial circular orbit. If the orbiting platform is rotated with a frequency ω_0 (measured according to a local clock) with respect to a set of local gyroscopes, the platform remains fixed with respect to locally determined (spherical) coordinate directions.[8] The tidal matrix is diagonal with

$$X^{\mathrm{S}} = [\phi/(1 - 3\phi)]\eta, \tag{5}$$

and

$$X^{\mathrm{LT}} \simeq \mp 2\phi^{-1/2}\psi\eta, \tag{6}$$

where $\eta_{rr} = \frac{3}{2}$ and $\eta_{\theta\theta} = 3$ are the only nonzero elements of η.[9] The ratio of the two effects is of the order of that of the rotation frequency of the source to the Keplerian frequency, just as for the two terms in the gyroscope precession. A (Schwarzschild) correction proportional to G^2 is not, of course, unexpected in a relativistic theory of gravitation. On the other hand, the Lense-Thirring correction is due to a coupling between the spin of the source and the *orbital* angular momentum of the platform. Thus the effect vanishes for a radial orbit along the axis of rotation. The specific form of this coupling for the equatorial orbit results in opposite effects for prograde and retrograde orbits. [The upper sign in Eq. (6) corresponds to a prograde orbit.]

To investigate the nature of this coupling further, we consider a geodesic polar orbit with constant r, $\theta = \omega\tau$, $\varphi = \Omega\tau$, and $t = \gamma\tau$. Here $\omega = \gamma\omega_0$ and

$$\gamma = (1 - 3\phi)^{-1/2}. \tag{7}$$

The nodes are dragged in the sense of the spin of the source at a rate (in units of proper time)

$$\Omega = 2\gamma A_{\mathrm{LT}}, \tag{8}$$

in accordance with the work of Lense and Thirring.[10] The gyroscope axes will be so chosen that initially ($\tau = 0$) the spatial directions are those of Cartesian axes for $J = 0$. The orbiting platform will be rotated with frequency ω_0 (according to the local clock) with respect to the gyroscopes so that the directions fixed in the platform correspond *approximately* to the (spherical) coordinate axes.[11] The tidal matrix with respect to these axes consists of a Newtonian part (just as before), a Schwarzschild part given by

$$X^{\mathrm{S}} = \gamma^2\varphi\zeta, \tag{9}$$

where the only nonzero elements of ζ are $\zeta_{\bar{r}\bar{r}} = \frac{3}{2}$ and $\zeta_{\bar{\varphi}\bar{\varphi}} = 3$, and a Lense-Thirring part with nonzero components

$$X_{\bar{\varphi}\bar{r}}^{\mathrm{LT}} = -2X_{\bar{r}\bar{\varphi}}^{\mathrm{LT}} = 9\alpha\gamma^2\phi^{-1/2}\psi\Delta, \tag{10}$$

and

$$X_{\bar{\theta}\bar{\varphi}}^{\mathrm{LT}} = X_{\bar{\varphi}\bar{\theta}}^{\mathrm{LT}} = 3\alpha\gamma^2\phi^{-1/2}\psi\sin(\omega_0\tau). \tag{11}$$

Here α and Δ are given by

$$\alpha = (1 - 2\phi)^{1/2}, \tag{12}$$

$$\Delta = \frac{1}{3\phi}[(1 + 2\phi)\cos(\omega\tau) - (1 - \phi)\cos(\omega_0\tau)]. \tag{13}$$

Thus for the polar orbit the rotation-dependent terms contribute only to the off-diagonal elements of the tidal matrix in contrast to the Schwarzschild effect. In this sense, the two effects are orthogonal for the polar orbit and parallel for the equatorial orbit, just as for the analogous effects in the precession of a gyroscope.

The Lense-Thirring term for the polar orbit is of a harmonic nature but its amplitude depends on the magnitude of $\omega_0\tau$. At early times, $\omega_0\tau \lesssim 1$, this effect is of the same order as for the equatorial orbit. However, at late times,[12] $1 \lesssim \omega_0\tau \ll \phi/\psi$, it could be larger by a factor of $\omega_0\tau$. This follows from Eq. (13) for $\omega_0\tau \ll \phi^{-1}$, since

$$\Delta \simeq \cos(\omega_0\tau) - \tfrac{1}{2}(1 + 2\phi)\omega_0\tau\sin(\omega_0\tau). \tag{14}$$

The appearance of the secular term, which is dominant after many revolutions, is due to a beat involving ω and ω_0. This same beat phenomenon, i.e., the deviation of orbital frequency according to a local clock from that determined by an inertial clock, is responsible for the Thomas precession. It would be of interest to measure this secular effect, thereby providing a new test of Einstein's theory of gravitation.

A complete assessment of the various requirements for the measurement of the new relativistic effects will not be attempted here except to point out that the same basic difficulty is encountered here as in the gyroscope experiment. To see this, one need only calculate the correction to the tidal matrix brought about by the deviation of the gyroscopes from ideal parallel transport due to extraneous torques. The analysis is simplified if we consider a local orthonormal tetrad $\bar{\lambda}_{(\alpha)}{}^{\mu}$, where $\bar{\lambda}_{(0)}{}^{\mu}=\lambda^{\mu}$ is a geodesic[13] and the spatial axes follow the transport law

$$D\bar{\lambda}_{(i)}{}^{\mu}/D\tau=\omega_i{}^{j}\bar{\lambda}_{(j)}{}^{\mu}, \tag{15}$$

with $\omega_{ij}(\tau)=-\omega_{ji}(\tau)$. It follows that the local motion of matter (referred to these axes) is influenced by a tidal acceleration term just as before, except that k must be replaced by κ,

$$\kappa_{ij}=\bar{k}_{ij}+\frac{d\omega_{ij}}{d\tau}+2\omega_{il}\gamma_j{}^{l}-\omega_{il}\,\omega_j{}^{l}, \tag{16}$$

where $\bar{k}$ is given by Eq. (1) with $\lambda_{(\alpha)}{}^{\mu}$ replaced by $\bar{\lambda}_{(\alpha)}{}^{\mu}$ and

$$\gamma_{ij}=\bar{\lambda}_{(i)}{}^{\mu}\lambda_{\mu i\nu}\bar{\lambda}_{(j)}{}^{\nu}. \tag{17}$$

In most applications—e.g., the oscillations of a spring in a fixed direction—the symmetric part of κ is of primary interest,

$$\kappa_{(ij)}=\bar{k}_{ij}+\omega_{il}\gamma_j{}^{l}+\omega_{jl}\gamma_i{}^{l}-\omega_{il}\,\omega_j{}^{l}. \tag{18}$$

Moreover, $\bar{k}$ differs from k by both secular and harmonic terms. A detailed examination (to first order in ω_{ij}) shows that, with neglect of the rotation of the central body and for constant ω_{ij},

$$\bar{k}_{ij}=k_{ij}+(k_{il}\,\omega_j{}^{l}+k_{jl}\,\omega_i{}^{l})\tau + \text{harmonic terms}, \tag{19}$$

where the harmonic terms are systematic and trace free. The quantities γ_{ij} have an amplitude of the order of ω_0; therefore, the amplitude of ω_{ij} must be kept well below $A_{\rm S}$ and $A_{\rm LT}$ in order to measure the Schwarzschild and the Lense-Thirring effects, respectively. But this is just the requirement necessary for the success of the Stanford gyroscope experiment.[14]

To sum up, the equations of motion of a physical system (e.g., a spring) with respect to a local inertial frame (introduced along the center of mass of a laboratory in orbit around the Earth) have their usual flat-space-time form, except for the presence of tidal accelerations which represent space-time curvature as "perceived" by the system. We have calculated these additional terms for a specific set of three orthogonal ideal gyroscopes representing the local *inertial* frame. The equations of motion with respect to any other set of three independent local axes can be obtained from the transformation of the corresponding equations in the inertial frame. The predictions of the theory concerning tidal accelerations experienced by the system depend on the axes used to define the local frame. The local axes may correspond, for instance, to real (i.e., nonideal) gyroscopes or to telescopes "fixed" on distant stars. How accurately should the local axes be defined along the orbit to allow a comparison of the theory with experiment? We have investigated this question only for a particular set of nonideal *gyroscopes*, but the answer appears to be general: Local gyroscopes must satisfy the same performance criteria as in the Stanford experiment. Therefore, a variant of the Stanford experiment is suggested: A precise gyroscope in polar orbit defines a "fixed" direction locally, along which the tidal acceleration acting on a spring is measured. The comparison of the technical feasibility of the two experiments is, however, beyond the scope of this paper.

We wish to record our gratitude to F. W. Hehl and other members of the Institute for Theoretical Physics (Cologne) for hospitality. This work was supported in part by the Deutsche Forschungsgemeinschaft, Bonn, by the Alexander von Humboldt Foundation, and by the Graduiertenförderung.

---

[1]B. Mashhoon, Astrophys. J. 216, 591 (1977), and references cited therein.

[2]Greek indices run from 0 to 3; Latin indices run from 1 to 3. Schwarzschild-like coordinates are used throughout, except when specified otherwise. The signature of the metric is +2 and $(0,1,2,3)=(t,r,\theta,\varphi)$.

[3]F. Tisserand, Compt. Rend. 75, 760 (1872), and 110, 313 (1890). See also G. Holzmüller, Z. Math. Phys. 15, 69 (1870). We are grateful to C. W. F. Everitt for bringing these references to our attention.

[4]H. Thirring, Phys. Z. 19, 33 (1918); J. Lense and

H. Thirring, Phys. Z. 19, 156 (1918); H. Thirring, Phys. Z. 22, 29 (1921). See also B. Mashhoon, F. W. Hehl, and D. S. Theiss, "On the Gravitational Effects of Rotating Masses: The Thirring-Lense Papers," to be published.

[5]C. W. F. Everitt, in *Experimental Gravitation*, edited by B. Bertotti (Academic, New York, 1973), p. 331; J. Lipa, W. M. Fairbank, and C. W. F. Everitt, *ibid.*, p. 361.

[6]For a circular earth orbit with r nearly equal to Earth's radius, $\phi \sim 6\times 10^{-10}$ and $\psi \sim 4\times 10^{-16}$. Hence $A_S \sim 4\times 10^{-13}$ rad s^{-1} and $A_{LT} \sim 3\times 10^{-14}$ rad s^{-1}.

[7]V. B. Braginsky and A. G. Polnarev, Pis'ma Zh. Eksp. Teor. Fiz. 31, 444 (1980) [JETP Lett. 31, 415 (1980)].

[8]This may be seen from the equivalence of our results with those given by L. G. Fishbone, Astrophys. J. 185, 43 (1973).

[9]For an equatorial circular orbit in a Kerr spacetime, $K = K^N + \chi_\pm K^N \eta$, where $\chi_\pm = (1 - 3\phi \pm 2\phi^{-1/2}\psi)^{-1} \times (1 \mp \phi^{-3/2}\psi)^2 \phi$. Hence $\chi\pm > 0$ for all circular orbits and diverges as the null orbit is approached. It is interesting to note that $\chi_\pm = \frac{1}{3}$ at the last stable circular orbit.

[10]See Ref. 4. Van Patten and Everitt have proposed an experiment to measure the effect of the dragging of inertial frames: R. A. Van Patten and C. W. F. Everitt, Phys. Rev. Lett. 36, 629 (1976), and Celest. Mech. 13, 429 (1976).

[11]As a reminder of this fact a bar is placed over indices referring to these tetrads. The nonzero components of $\Lambda_{(\bar{r})}{}^\mu$ and $\Lambda_{(\bar{\theta})}{}^\mu$ are given by $[\Lambda_{(\bar{r})}{}^r = \alpha$, $r\Lambda_{(\bar{\theta})}{}^\theta = \alpha\gamma$, $\alpha\Lambda_{(\bar{\theta})}{}^t = \gamma\phi^{1/2}$,

$$r\Lambda_{(\bar{r})}{}^\varphi = \alpha\phi^{-3/2}\psi[\cot(\omega\tau) - \csc(\omega\tau)\cos(\omega_0\tau)],$$

and

$$r\Lambda_{(\bar{\theta})}{}^\varphi = \alpha\phi^{-3/2}\psi[\csc(\omega\tau)\sin(\omega_0\tau) - \beta],$$

where $\beta = \gamma(\alpha^2 - \phi\alpha^{-2})$. The components of $\Lambda_{(\bar{\varphi})}{}^\mu$ may be determined from the orthonormality conditions. It is important to note that the rotation of the platform by frequency ω_0 only helps simplify the final result and is not the cause of the *main* effects considered in this paper.

[12]The angular momentum of the source has been taken into account only to first order: therefore, the linear results may break down for $\tau \gtrsim \Omega^{-1}$.

[13]To ensure this, a sufficiently drag-free laboratory system is necessary just as in the gyroscope experiment.

[14]It should be pointed out that our results differ from those of Ref. 7 in several important respects: (i) The Schwarzschild effect is not mentioned in Ref. 7, (ii) the amplitude of the rotation-dependent effect is not given correctly since the secular term is absent, and (iii) the opposite conclusion is implied regarding the question of measurability of the relativistic effects.

Influence of Dissipation on Quantum Coherence

A. J. Bray and M. A. Moore
Department of Theoretical Physics, The University, Manchester, M13 9PL, United Kingdom

(Received 30 June 1982)

A quantum mechanical particle which moves in a symmetric double-well potential, and whose interaction with the environment is described in the classical regime by a phenomenological friction coefficient η, is considered. It is shown that, provided $\eta q_0^2/\hbar$ exceeds a critical value of order unity ($\pm q_0$ are the locations of the potential minima), the mean rate of tunneling between the degenerate minima decreases with temperature, leading at $T = 0$ to spontaneous symmetry breaking.

PACS numbers: 05.30.-d, 03.65.Bz, 73.40.Gk

There has been recent interest[1,2] in the influence of dissipation on quantum *tunneling* out of a metastable state. Here we consider the related problem of quantum *coherence*. Specifically we consider a particle of mass M moving in a symmetric double-well potential $V(q) = V(-q)$ which has minima at $q = \pm q_0$ and a local maximum at $q = 0$, and whose classical equation of motion is $M\ddot{q} + \eta\dot{q} = -dV/dq + F_{\text{ext}}(t)$. We limit our considerations to temperatures small compared to the frequency $\omega_0 = [M^{-1}V''(q_0)]^{1/2}$ of small oscillations around one of the minima: $k_B T \ll \hbar\omega_0$, the limit in which thermal activation over the barrier can be neglected compared with quantum tunneling. Further, in the limit $\hbar\omega_0 \ll \Delta V = V(0) - V(q)$, one can truncate to the lowest two states ψ_s, ψ_a with energies E_s, E_a, respectively. If the system is prepared at $T = 0$ and time $t = 0$ in the state $\psi_L = (\psi_a + \psi_s)/\sqrt{2}$ representing a wave packet localized in the left-hand well, the amplitude for being in the left-hand well at time t oscillates with frequency $\Delta_0/\hbar = (E_a - E_s)/\hbar$. This is the phenomenon of "quantum coherence." At finite temperatures this oscillation