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Painlevé Conjecture Revisited

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The discovery of new integrable two-dimensional Hamiltonian systems is reported. The analytic structure of the solutions makes necessary the generalization of the Painlevé conjecture, a widely used integrability criterion. Such a generalization is presented, which the authors believe should replace the usual conjecture for two-dimensional Hamiltonian systems. It is indeed compatible with all the systems already found and, in addition, leads to still new integrable cases.

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The extreme rarity of integrable dynamical systems makes the quest for them all the more exciting. Till very recently, the search for such systems was based mainly on intuition and on extensive (and expensive!) numerical investigations. However, in the last few years the works of Ablowitz, Ramani, and Segur (ARS)[1-3] have provided us with a most useful tool for the identification of integrable systems. They have conjectured, and, subsequently abundantly verified, that a dynamical system described by a differential system or a partial differential equation is integrable whenever the solutions have the Painlevé property,[4] namely, that their only movable singularities are poles.

The method has met with an undeniable success as it has guided the search for integrals of motion for dynamical systems of physical interest. In the case of the Lorenz system, Segur was able to identify integrable cases with the help of the Painlevé property and proceeded to calculate the integrals of motion.[5] The Henon-Heiles Hamiltonian is another example. The analysis by Chang *et al.* has resulted in three cases having the Painlevé property.[6] One corresponds to a well-known separable case.[7] In the second case the integral of motion was calculated by Greene.[8] The recent derivation of the second integral for the third case independently by Hall[9] and Grammaticos, Dorizzi, and Padjen[10] fully confirmed the predictive character of the Painlevé property.

At this point one can ask a most pertinent question. Do all integrable systems have the Painlevé property? This has been tacitly assumed and actually verified on a host of integrable cases to date.[6,11,12] However, some recent results, if interpreted in a narrow-minded way, would tend to suggest the contrary. The aim of this paper is to present these new integrable systems and to show how the conjecture concerning the Painlevé property must be extended so as to accomodate these new cases of integrability. The need to generalize the ARS conjecture to allow for a class of singularities less restrictive than was previously thought can be seen from the following remarks:

(a) One-dimensional Hamiltonian systems are integrable by definition. Clearly, the integrability is independent of the analytical properties of the solution in the complex time plane. Logarithmic cuts or even essential singularities which can easily be built into the Hamiltonian do not hinder integrability. (b) It has long been known that the Painlevé property should be extended in order to include a global change of variables, provided this single change transforms the singularities of *all* the solutions into poles.[3] This was considered a minor generalization of the Painlevé property. However, the conclusive case was the discovery of (c) an integrable Hamiltonian system in two dimensions, for a mass-one particle, with a quintic polynomial potential:

$$V = y^5 + y^3x^2 + \tfrac{3}{16}yx^4.$$

This system has a second integral of motion in addition to the total energy. This integral is

$$C = -y\dot{x}^2 + x\dot{x}\dot{y} + \tfrac{1}{2}x^2y^4 + \tfrac{3}{8}y^2x^4 + \tfrac{1}{32}x^6.$$

An analysis of the complex time (t) singularities of the system shows that they start as $(t-t_0)^{-2/3}$. This, in itself, need not be a problem. It might, in principle, be accomodated by remark (b) above, had the cube of the solution had a pure double pole. However, the expansion of the solution around the singularity contains *all* powers of $(t-t_0)^{1/3}$. This is in direct contradiction with the ARS conjecture as previously meant and leads us to accept a wider class of movable singularities.

In order to find a new integrability criterion, we start by excluding one-dimensional systems and fully separable cases which are equivalent to them. Analytic properties do not play any role in these cases. For two-dimensional systems, we will say that a system possesses the "weak Painlevé property" whenever the solution, in the neighborhood of a singularity at t_0, can be expressed as an expansion in powers of $(t-t_0)^{1/r}$. Here r is an integer which must be the "natural" one, given solely by the leading behavior of the singularity. The new conjecture is now that integrable systems have the weak Painlevé property. At this point we emphasize that one can recover poles neither by raising the solution to the rth power, nor by a change of independent variable, since the singularity is movable. This is a genuine weakening of the Painlevé property. However, singularities of logarithmic or irrational power type are still conjectured to be incompatible with integrability. For example,

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + ax^3 + by^3 + cxy$$

has a logarithmic singularity for $c \neq 0$ and is only integrable for $c = 0$.

For the more than two-dimensional cases, one can only surmise that the need for the weakening of the Painlevé concept will become less and less felt with increasing dimensionality. Already in three dimensions there does not seem to be any counterexample to the original Painlevé conjecture. However, results in these cases are scant and any systematic search prohibitively involved. For infinite dimensions, that is, partial differential equations, it seems safe to put forward that the conjecture will remain valid in its original form.

Now that the conjecture has been reformulated, one must question its predictive character. Can one use it to find new integrable systems? The following will show that the answer to this question is affirmative.

A first example is offered by a family of Hamiltonians to which the example in (c) belongs. These Hamiltonians describe a mass-one particle in two dimensions subject to a homogeneous polynomial potential of the form

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} C_{n-k}{}^{k} x^{2k} y^{n-2k}.$$

One can actually show that all these Hamiltonians are integrable, the second integral of motion being

$$C_n = -y\dot{x}^2 + x\dot{x}\dot{y} + \sum_{k=0}^{[(n-1)/2]} 2^{n-2k-1} C_{n-k-1}{}^{k} x^{2k+2} y^{n-2k-1},$$

that is,

$$C_n = -y\dot{x}^2 + x\dot{x}\dot{y} + x^2 V_{n-1}.$$

One has found this family of homogeneous polynomials by attempting to satisfy the weak Painlevé property in the easiest way through some simplifying assumptions. This leads to the following partial differential equation for V:

$$2y\frac{\partial^2 V}{\partial x \partial y} + 3\frac{\partial V}{\partial x} + x\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) = 0.$$

The same partial differential equation is obtained if one looks directly for an integral of motion quadratic in the velocities. This shows that any linear combination of the V_n's also leads to an integrable Hamiltonian. Actually the Henon-Heiles potential, in the case integrated by Greene,[8] obeys just this equation, and is a linear combination of V_3, V_2, and V_1, up to a suitable linear transformation of variables. The derivation of this equation, together with its general solution, which we have obtained, will be presented in a future publication.

The family of integrable potentials we have just presented are not the only ones. The class of integrable systems can be further expanded if one relaxes the simplifying assumptions we alluded to previously. However, the calculations become hopelessly intricate unless one limits oneself to some low-order polynomial potentials.

We have fully investigated the case of cubic polynomial potentials, and have been able to find *all* weak Painlevé cases. These include all the known integrable cases[8,9] and, in addition, a unique (up to rotation and complex conjugation) new case,

$$V = y^3 + \tfrac{1}{2} y x^2 + i x^3/6\sqrt{3},$$

for which the integral has been found:

$$C = \dot{y}^4 + 2\dot{y}^2\dot{x}^2 - 2i\dot{y}\dot{x}^3/\sqrt{3} + (4y^3 + 2yx^2 - ix^3/3\sqrt{3})\dot{y}^2 + (i\sqrt{3}yx^2 + x^3)\dot{x}\dot{y}$$
$$+ (4y^3 - 2i\sqrt{3}y^2x - ix^3/3\sqrt{3})\dot{x}^2 + 4y^6 + 4y^4x^2 + ix^3y^3/3\sqrt{3} + 5y^2x^4/4 + ix^5y/6\sqrt{3} + x^6/54.$$

One can show directly that this exhausts all third-order potentials allowing integrals which are polynomial of order four or less in the velocities.

At order four, we found two new integrable potentials by looking for the weak Painlevé property. One is the $n=4$ case of the family described above,

$$V_4 = 16y^4 + 12x^2y^2 + x^4,$$

for which the integral is

$$C_4 = -y\dot{x}^2 + x\dot{x}\dot{y} + 8x^2y^3 + 4x^4y.$$

The other is

$$V = 8y^4 + 6x^2y^2 + x^4,$$

for which the integral is

$$C = \dot{x}^4 + (24x^2y^2 + 4x^4)\dot{x}^2 - 16x^3y\dot{x}\dot{y} + 4x^4\dot{y}^2 + 4x^8 + 16x^6y^2 + 16x^4y^4.$$

We do not claim that this exhausts the weak Painlevé, nor integrable, cases for fourth-order potentials.

We therefore conjecture that, for two-dimensional Hamiltonian systems, the appropriate generalization of the Painlevé property we introduced, rather than the original form of this property, is characteristic of integrability. Moreover, our results constitute ample evidence that this generalized Painlevé property is a most useful tool in the search for integrable dynamical systems.

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