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Successive Higher-Harmonic Bifurcations in Systems with Delayed Feedback

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The scheme of bifurcations in systems described by a certain class of delay-differential equations is investigated in detail. It is shown that higher-harmonic oscillating states appear successively in the course of transition to developed chaos. First-order transitions between these states account for the frequency-locked anomaly recently observed by Hopf *et al.* in a hybrid optical bistable device.

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Among dynamical systems capable of displaying chaotic behavior, systems with delayed feedback are of interest since their time evolution is determined by a concurrence of a discrete step which tends to induce chaos and a continuous step which tends to smear it. A physical example of such systems is an optical cavity filled with a nonlinear dielectric medium and irradiated with a laser light of constant intensity. Several authors investigated the phenomenon that the transmitted light from such a cavity, a part of which is fed back to the medium by mirrors, displays bistable behavior[1,2] and chaotic behavior[3-5] under suitable conditions. Other examples are found in neurobiology[6] and ecology.[7]

Recently, Hopf and others observed, in a system equivalent to a nonlinear optical cavity, a novel phenomenon they named frequency-locked anomaly[8]: In the course of transition from periodic states to developed chaos, there appear chaos with traces of periodic structure and its higher harmonics successively, being accompanied by discontinuous transitions between them. In this Letter we investigate in detail the scheme of bifurcations on a delay-differential equation which models the dynamics of the system and point out that the above phenomenon is of universal character in a certain class of systems with delayed nonlinear feedback.

The delay-differential equation we investigate is of the form

$$dx(t)/dt = -x(t) + f(x(t-t_R);\ \mu), \tag{1}$$

where μ is the bifurcation parameter. The second term on the right side represents nonlinear feedback with delay t_R. For an optical cavity, $f(x;\ \mu)$ in Eq. (1) takes the special form[3]

$$f(x;\ \mu) = \pi\mu[1 + 2B\cos(x - x_0)], \tag{2}$$

where μ is proportional to the power of the incident light, and B (<1) represents the dissipation of the electromagnetic field in the cavity. The delay t_R is the time required for light to make a round trip in the cavity.

We first solved Eq. (1) with (2) numerically, increasing μ slowly ($d\mu/dt < 5\times10^{-7}$) from zero. The other parameters were fixed at $B = 0.5$, $x_0 = -\pi/2$, and $t_R = 40$ so that they can reproduce the experimental condition of Hopf *et al.* The results are summarized as follows. As μ exceeds the critical value $\mu_A \cong 0.376$, a square-wave solution of period T_0 ($\cong 2t_R$) appears after the Hopf bifurcation of a stationary solution. With further increase of μ, this square-wave solution undergoes a sequence of bifurcations with its period doubling itself successively. As μ reaches the Feigenbaum point $\mu_F \cong 0.696$, the solution becomes chaotic. Coarsely seen, however, this chaotic solution is still square-wave-like with period T_0, as is shown in Fig. 1(a). In fact it appears that a small fluctuation is superposed on the square-wave solution of period T_0. As μ exceeds $\mu_B \cong 0.775$, this coarsely square-wave-like solution also becomes unstable and successive bifurcations of a new type appear: The period of the solution changes discontinuously like $T_0 \to T_0/3 \to T_0/5 \to T_0/7$ with the increase of μ (Fig. 1). Correspondingly, the position of the highest peak in the power spectrum shifts discontinuously like ω_0 ($= 2\pi/T_0 \cong \pi/t_R$) $\to 3\omega_0 \to 5\omega_0 \to 7\omega_0$. The solution of period T_0/n (n: odd integer) is thus regarded as a higher harmonic of the fundamental solution with period T_0. These results, including the critical values of μ, reproduce the frequency-locked anomaly observed by Hopf *et al.* fairly well.

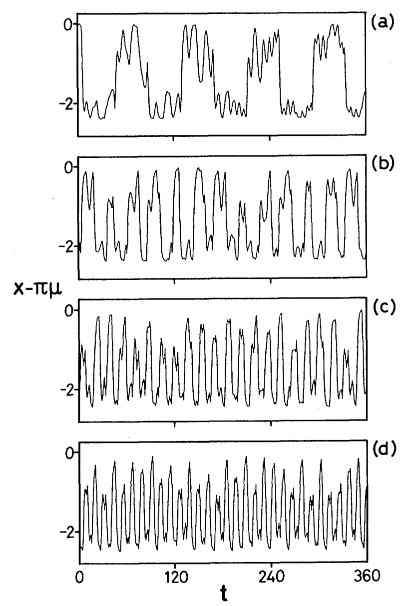

FIG. 1. Successive higher-harmonic bifurcations for $B = 0.5$, $x_0 = -\pi/2$, and $t_R = 40$: (a) fundamental ($\mu = 0.770$), (b) third harmonic ($\mu = 0.778$), (c) fifth harmonic ($\mu = 0.780$), and (d) seventh harmonic ($\mu = 0.822$).

The transitions between the harmonic solutions are of first order, being accompanied by a hysteresis for slow increase and decrease of μ. This means that several harmonic solutions coexist as stable attractors for one value of μ. Figure 2 shows an example of the domains of μ in which the harmonic solutions exist and the hysteresis between them. The ordinate Ω represents the mean frequency of the corresponding solution, so that $\Omega = n$ holds for an ideal nth harmonic. The branch of every harmonic consists of (purely) periodic and chaotic parts. With increase of n, the domain of the harmonic becomes narrower and shifts towards the higher side of μ. In some cases, the destination of transition is not unique; it depends sensitively on the velocity of scanning of μ, especially in the downward transition.

The degree of the harmonic solutions realized by increasing μ has its limit. (In Fig. 2, e.g., it does not go beyond seven.) When μ is further increased, the harmonic solution of the maximum degree also becomes unstable, changing to developed chaos showing totally complicated time evolution. If μ is fixed at a certain value slightly larger than μ_B and t_R is increased, on the other hand, the degree of the harmonics increases without limit, going through every odd number, as is shown in Fig. 3. In this case a first-order transition occurs every time t_R is increased by a definite amount. The mean fre-

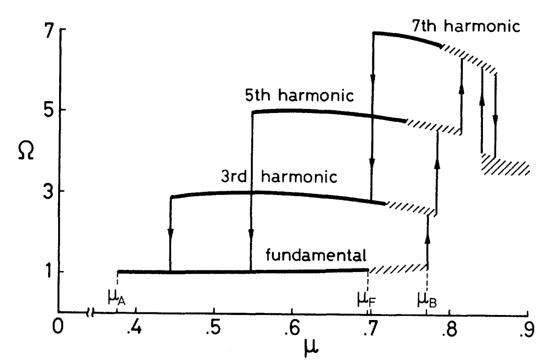

FIG. 2. Domains of several harmonic solutions and hysteresis between them. The parameter values are the same as those in Fig. 1. The solid lines and the shaded parts indicate periodic and chaotic solutions, respectively.

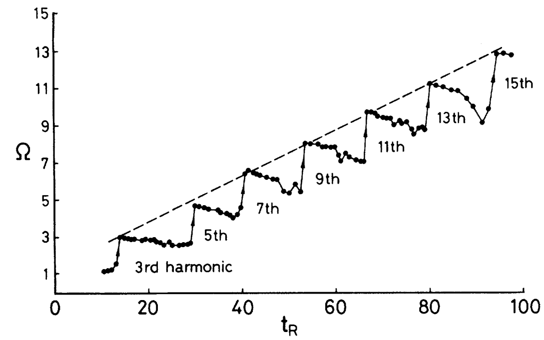

FIG. 3. Successive higher-harmonic bifurcations obtained by increasing t_R for $B = 0.5$, $x_0 = -\pi/2$, and $\mu = 0.82$.

quency Ω does not agree with the degree of the harmonics precisely, because the solutions are not purely periodic, but involve chaotic components.

Figure 4 shows the existence domain of several harmonic solutions in the parameter space (t_R, μ). The boundary of each domain has two common asymptotes, $\mu = \mu_A$ and $\mu = \mu_B$. For $\mu_A < \mu < \mu_B$, a number of domains coexist, overlapping each other. The domains intersect a line of constant μ at regular intervals, so that the number of the coexisting harmonics is proportional to t_R. Figures 2 and 3 have been obtained by scanning parameters along the lines l_a and l_b, respectively, in Fig. 4. Above $\mu = \mu_B$ the domains of harmonics coexist with the domain of developed chaos in a complicated manner, though not illustrated in Fig. 4, so that the ninth and the eleventh harmonics are not realized by increasing μ along l_a, but rather the system breaks into a developed chaos.[9] For the fundamental solution with small t_R, there is no clear-cut boundary between the coarsely periodic and the totally chaotic states.

The origin of the behavior mentioned above is understood as follows. For large t_R, Eq. (1) is formally approximated by the difference equation

$$x(t) = f(x(t - t_R); \mu). \tag{3}$$

If the map $x_{p+1} = f(x_p; \mu)$ has a solution of period 2 satisfying $x_1 = f(x_2; \mu)$ and $x_2 = f(x_1; \mu)$, then it is clear that Eq. (3) has the solution $x(t) = x_1$ for $t \in I^{(2p)}$ and $x(t) = x_2$ for $t \in I^{(2p+1)}$, where $I^{(p)} = (t^{(p)}, t^{(p+1)}]$ with $t^{(p)} = pt_R$. The period of this solution is $2t_R$, so that it corresponds to the fundamental square-wave solution of Eq. (1). It is not the unique solution of Eq. (3), however. Divide the section $I^{(p)}$ into n subsections $I_1^{(p)}$

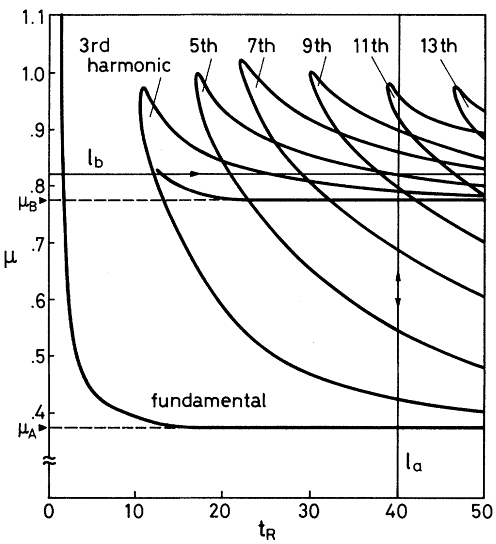

FIG. 4. Domains of the harmonic solutions in the parameter space (t_R, μ). The other parameters are fixed at $B = 0.5$ and $x_0 = -\pi/2$.

$= (t^{(p)}, t^{(p)} + t_1]$, $I_2^{(p)} = (t^{(p)} + t_1, t^{(p)} + t_2], \ldots, I_n^{(p)} = (t^{(p)} + t_{n-1}, t^{(p+1)}]$ (n: odd), where $\{t_k\}$ is an arbitrary sequence of t satisfying $0 < t_1 < t_2 < \cdots < t_{n-1} < t_R$. Then, the general solution of Eq. (3) is given by the following: $x(t) = x_1$ for $t \in I_{2k-1}^{(2p)}$ or $I_{2k}^{(2p+1)}$ and $x(t) = x_2$ for $t \in I_{2k}^{(2p)}$ or $I_{2k-1}^{(2p+1)}$, where $1 < k < (n+1)/2$. We call this solution the fissured solution of the nth degree. We remark that there are infinite numbers of fissured solutions of the same degree, because the number of ways of choosing $\{t_k\}$ is infinite. It might be considered that all of these fissured solutions have their counterparts in the solutions of Eq. (1), or in other words, that all of the fissured solutions can be approximate solutions of Eq. (1). This is not the case, however, for the following reason. Equation (1) can be rewritten in the integral form

$$x(t) = \int_{-\infty}^{t} e^{-(t-s)} f(x(s - t_R); \mu)\, ds, \tag{4}$$

so that the value of x at t is determined by a weighted average of $f(x)$ in the vicinity of $t - t_R$. The case is different in Eq. (3), in which $x(t)$ is determined by only $f(x)$ at $t - t_R$. As a result of this averaging, only the fissured solution having some regularity in the intervals between t_p and t_{p+1}, i.e., the higher harmonic of the fundamental square-wave solution, survives as an approximate solution of Eq. (1). On the other hand, when the division interval t_R/n becomes smaller than the averaging width of Eq. (1), which is of the order of unity, the fissure structure is also smeared out and can no longer be an approximate solution of Eq. (1). Thus, n has an upper limit of the order of t_R. This is the reason why the number of the coexisting harmonic solutions is proportional to t_R.

The above consideration is valid for any delay-differential equation of the form (1), provided the corresponding difference equation (3) exhibits period-doubling bifurcations. To confirm this, we made the same numerical simulation for the following four functions $f(x;\mu)$: (i) $\mu x(1-x)$, (ii) $\mu x(1-x^2)$, (iii) $\mu x e^{-x}$, and (iv) $\mu x(1+x^4)^{-1}$. As was expected, structures of parameter domains which are topologically equivalent to Fig. 4 have been found for all of these functions. A remarkable fact is that the maximum degree of the harmonic solutions realized by increasing μ for fixed t_R, $n_{\max}$, is about the same in all cases. For example, we found $n_{\max}=3-5$ for $t_R=20$ and $n_{\max}=7$ for $t_R=40$ in all cases. This tells us that there may exist some quantitative universality in the parameter dependence of the number of the coexisting higher-harmonic solutions. Further investigations will be reported elsewhere.

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[1]H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. 19, 1135 (1976), and Opt. Eng. 19, 463 (1980), and references cited therein.

[2]*Optical Bistability*, edited by C. M. Bowden, M. Ciftan, and H. R. Robl (Plenum, New York, 1981).

[3]K. Ikeda, Opt. Commun. 30, 257 (1979); K. Ikeda, H. Daido, and O. Akimoto, Phys. Rev. Lett. 45, 709 (1980).

[4]K. Ikeda and O. Akimoto, Phys. Rev. Lett. 48, 617 (1982).

[5]H. M. Gibbs, F. A. Hopf, D. L. Kaplan, and R. L. Shoemaker, Phys. Rev. Lett. 46, 474 (1982).

[6]J. S. Griffith, *Mathematical Neurobiology* (Academic, London, 1971).

[7]R. May, in Proceedings of the Les Houches Summer School on Chaotic Behavior of Deterministic Systems, July 1981 (to be published).

[8]F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker, Phys. Rev. A 25, 2172 (1982).

[9]In experiments, lockings to the ninth to thirteenth harmonics have been observed even by scanning μ for fixed t_R. The reason for this disagreement with the numerical results is not clear.

Bell's Theorem as a Nonlocality Property of Quantum Theory

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Bell's theorem is formulated as a nonlocality property of quantum theory itself, with no explicit or implicit reference to determinism or hidden variables. A recent Letter on this subject is discussed.

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In a recent Letter[1] related to Bell's theorem[2] Fine proved several propositions, and asserted the following conclusion: "Proposition (2) shows that, despite appearances, no significant generality is achieved by those derivations of the Bell/CH inequalities that dispense with explicit reference to hidden variables and/or determinism: The assumptions of such derivations imply the existence of deterministic hidden variables for any experiment to which they apply."

This conclusion consists of two assertions, which must be distinguished. The second is meant to be a rephrasing of proposition (2), and, as such, is technically correct. However, it is misleading because of two semantic irregularities: (1) Fine leaves the word "local" out of his name "deterministic hidden-variables models." Usually this word is inserted to remind the reader that the models in question have an important factorization property that normally arises from the idea that the deterministic hidden variables are separated into two local parts, each of which determines those results of the experiment that occur in one of two separated regions. (2) Fine leaves the word "model" out of the rephrasing. This creates the impression that what was proved

