Evolution of Packets of Water Waves for Finite-Wave-Number Perturbations

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The evolution of packets of water waves with wave amplitudes modulated in horizontal directions is investigated. The analysis is based on the Davey-Stewartson equations which are valid for water of finite depth. It is shown that for finite wave numbers l of the transverse disturbances an instability occurs: The maximum growth rates are calculated and their physical implications are discussed.

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Various nonlinear evolution equations for surface waves in water under the action of gravity are of interest. For shallow water, a Korteweg-de Vries (KdV) type equation was derived[1] for nearly one-dimensional long waves. It has interesting, one-dimensionally stable, soliton solutions. The validity of this approach is limited by the condition that the height φ_0 of the soliton should be much smaller than the depth h of the water ($\varphi_0/h \ll 1$). For deep water, a nonlinear Schrödinger equation was derived[2] which allows for envelope solitons. Here, in contrast to the KdV soliton, the envelope of a wave train (with wave number k and frequency ω) is slowly modulated. The envelope-soliton solutions of the nonlinear Schrödinger equation are valid for $\varphi_0/h \ll k\varphi_0 \ll 1$. Recently, interest has been growing in the intermediate region where kh is finite. Davey and Stewartson[3] have derived a coupled set of equations which are valid for irrotational fluids in the limit $(\varphi_0/h)^{1/2} \ll kh \ll h/\varphi_0$. In the short-wave-number region, these equations reduce to the nonlinear Schrödinger equation whereas in the long-wave limit $[(\varphi_0/h)^{1/2} \ll kh \lesssim 1]$ the completely integrable Anker-Freeman equations[4] result. Besides these small-amplitude theories, recently[5] also the problem of finite-amplitude solutions has been attacked on the basis of the full set of hydrodynamic equations and boundary conditions.

All the model equations allow us to study the two-dimensional evolution of the corresponding soliton solutions. When the stability of one-dimensional envelope solitons is considered[6] a very interesting result occurs: For large kh values, the plane solutions become two-dimensionally unstable, whereas for $kh \simeq 1$ they are stable. This leads to the question of when exactly the transition from unstable to stable behavior occurs and to the problem of estimating the growth rates in the unstable region. This Letter is devoted to the second problem, whereas the first question may be answered[6] by $kh \simeq 1.363$. So far, except for the case of the cubic nonlinear Schrödinger equation, the complete dependence of the growth rates on the perturbation characteristics (wave number l) is not known. In all theories, a small-l expansion is performed which does not allow one to calculate the maximum growth rates and cutoffs. But the latter are needed to predict lifetimes and observability. In the following we shall derive the l dependence of the growth rates and present the maximum growth rates as functions of kh.

We describe the wave envelopes by the Davey-Stewartson equations[7] in the form

$$i\partial_\tau A + \lambda\partial_\xi^{\,2} A + \mu\partial_\eta^{\,2} A = \nu|A|^2 A + \nu_1 A Q, \tag{1}$$

$$\lambda_1\partial_\xi^{\,2} Q + \mu_1\partial_\eta^{\,2} Q = \kappa_1\partial_\eta^{\,2}|A|^2. \tag{2}$$

Here, A corresponds to the first-order surface elevation φ $[g\varphi = i\omega A\exp(ikx - i\omega t) + \text{c.c.}$, where g is the acceleration due to gravity$]$ and Q is related to the velocity potential. The longitudinal and transverse (slow) space coordinates are denoted by ξ and η, and τ is the (slow) time coordinate. The coefficients λ, μ, ν, ν_1, λ_1, μ_1, and κ_1 are given[8] in Ref. 3. It should be noted that the coefficient ν approaches 0 as kh approaches 1.363; therefore the discussion can be split into two parts: When $kh < 1.363$ $(\nu < 0)$, the *stability* of plane solitons has been shown[6] already by investigating the Anker-Freeman equations[4]; the *instability* for $kh > 1.363$ $(\nu > 0)$ will be discussed in this paper.

For large values of kh, the system (1) and (2) reduces to the cubic nonlinear Schrödinger equation

$$i\partial_\tau A - \tfrac{1}{8}g^{1/2}k^{-3/2}\partial_\xi^{\,2} A + \tfrac{1}{4}g^{1/2}k^{-3/2}\partial_\eta^{\,2} A - 2k^{7/2}g^{-1/2}|A|^2 A = 0, \tag{3}$$

which we will consider as a reference system.

The stationary one-dimensional soliton solutions of Eqs. (1) and (2) [or Eq. (3)] contain a free parameter A_0 and can be written in the form

$$A = A_0 \operatorname{sech}[A_0(-\nu/2\lambda)^{1/2}\xi] \exp(-i\nu A_0^2\tau/2), \quad (4)$$

$$Q = 0. \quad (5)$$

The maximum elevation of the wave is defined by $\varphi_0 \equiv \omega A_0/g$. Perturbing A,

$$A = [G(\xi) + u + iv] \exp(-i\nu A_0^2\tau/2),$$

where

$$G(\xi) = A_0 \operatorname{sech}[A_0(-\nu/2\lambda)^{1/2}\xi],$$

and Fourier analyzing in the transverse direction (wave number l) we find for the exponential growth rate γ_l the complementary variational principles[9]

$$\gamma_l^2 = \sup_{\psi \text{ odd}} \frac{-\langle\psi|H_+H_-H_+|\psi\rangle}{\langle\psi|H_+|\psi\rangle} \quad (6)$$

and

$$\gamma_l^2 = \inf_{\psi\in M} \frac{-\langle\psi|H_-H_+H_-|\psi\rangle}{\langle\psi|H_-|\psi\rangle}. \quad (7)$$

Here,

$$H_- = \lambda\partial_\xi^2 - \mu l^2 - 3\nu G^2 + \nu A_0^2/2 + 2\kappa_1\nu_1 l^2 G\nabla^{-2}G, \quad (8)$$

$$H_+ = \lambda\partial_\xi^2 - \mu l^2 - \nu G^2 + \nu A_0^2/2, \quad (9)$$

$$\nabla^{-2} = (\lambda_1\partial_\xi^2 - \mu_1 l^2)^{-1}, \quad (10)$$

and M is the set of odd functions ψ with $\langle\psi|H_-|\psi\rangle > 0$. Several comments are in order: First, we are using complementary variational principles in order to find upper and lower bounds for the growth-rate curves by standard numerical procedures. Secondly, the formulations (6) and (7) are based on definiteness properties of H_+ and H_-. However, H_+ is only positive definite for $2\mu l^2/\nu < A_0^2$. Therefore, our calculations cannot cover the whole l range; but, as the results will show, the maximum growth rates occur within this region: Beyond that limit, numerical solutions of the eigenvalue problem are necessary, as in the case of the nonlinear Schrödinger equation. Finally, for large kh values, the Davey-Stewartson equations (1) and (2) reduce to the cubic nonlinear Schrödinger equation (3), and expressions (6) and (7) reduce to the corresponding variational principles for the Schrödinger equation. To test our procedure, we have evaluated this special case first. When new coordinates for time ($t = -\tau/2\omega$), and space ($x = \sqrt{2}\,\xi k^{1/2}/g^{1/2}$; $y = \sqrt{2}\,\eta k^{1/2}/g^{1/2}$) are used and the variable A is changed to $\tilde{A} = 2k^2 A$, Eq. (3) simplifies to

$$i\partial_t\tilde{A} + \tfrac{1}{2}\partial_x^2\tilde{A} - \partial_y^2\tilde{A} + |\tilde{A}|^2\tilde{A} = 0. \quad (11)$$

We write its soliton solution as

$$\tilde{A} = \sqrt{2}\,\tilde{A}_0 \operatorname{sech}(\sqrt{2}\,\tilde{A}_0 x). \quad (12)$$

For the Schrödinger equation, the operators (8) and (9) are

$$H_+ = -\tfrac{1}{2}\partial_x^2 - l^2 - 2\tilde{A}_0^2 \operatorname{sech}^2 x + \tilde{A}_0^2, \quad (13)$$

$$H_- = -\tfrac{1}{2}\partial_x^2 - l^2 - 6\tilde{A}_0^2 \operatorname{sech}^2 x + \tilde{A}_0^2. \quad (14)$$

We have evaluated the variational principles (6) and (7) and find excellent agreement with the eigenfunction solutions of Saffman and Yuen.[6] The results are shown in Fig. 1; the growth-rate curve is plotted broken for $l^2 > \tilde{A}_0^2$ since then H_+ changes its definiteness properties and our variational approach is strictly valid only for positive definite operators H_+. The straight broken line (valid for $l^2 \ll \tilde{A}_0^2$) is the asymptotic solution of Zakharov and Rubenchik[10] ($\gamma_l^2 \simeq \frac{4}{3}\tilde{A}_0^2 l^2$) which we reproduce by inserting $\psi = \operatorname{sech}^2 x \tanh x$ and $\psi = x \operatorname{sech} x$ into (6) and (7), respectively.

For the Davey-Stewartson equations both variational principles (6) and (7) were evaluated numerically by a von Mises procedure with a Wieland iteration. With use of a finite number of test functions $\psi_{i/2}$ ($i = 1, \ldots, n$) the problem reduces to finding the largest eigenvalues of finite n-dimensional matrices. The main difficulty occurs when the matrix elements of the ∇^{-2} opera-

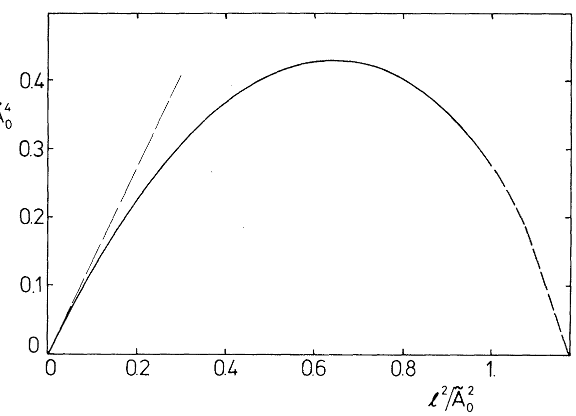

FIG. 1. Growth rate γ_l vs transverse wave number l for the cubic nonlinear Schrödinger equation (11). The straight broken line is the asymptotic behavior for small l.

tor in H_- have to be determined. Using

$$\psi_{i/2} = \partial_\xi \operatorname{sech}^{(i/2)}[A_0(-\nu/2\lambda)^{1/2}\xi], \quad i=1,\ldots,20, \tag{15}$$

we have analytically traced back the most difficult elements

$$\langle\psi_i|H_+ G\nabla^{-2} G H_+|\psi_j\rangle,$$

$$\langle\psi_i|\operatorname{sech}\xi\,\nabla^{-2}\operatorname{sech}\xi\,H_+\operatorname{sech}\xi\,\nabla^{-2}\operatorname{sech}\xi|\psi_j\rangle$$

to the basic ones for $i,j=1,2,\frac{1}{2},1\frac{1}{2}$. The elements containing half-integer indices were computed numerically whereas the rest can be evaluated analytically,

$$\langle\psi_1|\nabla^{-2}|\psi_1\rangle = -(4/l\mu_1^{1/2}\lambda_1^{1/2})\zeta(2,\tfrac{1}{2}(\mu_1^{1/2}l/\lambda_1^{1/2}+1)), \tag{16}$$

$$\langle\psi_2|\nabla^{-2}|\psi_1\rangle = -(\pi/\lambda_1)[D(\tfrac{1}{2}(\mu_1^{1/2}l/\lambda_1^{1/2}+1)) - D(\mu_1^{1/2}l/2\lambda_1^{1/2}) - 1], \tag{17}$$

$$\langle\psi_2|\nabla^{-2}|\psi_2\rangle = (4\lambda_1^{1/2}/l\mu_1^{1/2})\int_0^\infty d\sigma\,\exp(-\mu_1^{1/2}l\sigma/\lambda_1^{1/2})[\sinh^{-2}(\sigma)(1-\sigma/\tanh\sigma)], \tag{18}$$

where ζ is Riemann's zeta function and D is the digamma function.

A typical result is shown in Fig. 2. Here we have plotted the growth rate γ_l as a function of l/k for various parameters kh and $\varphi_0/h = 0.025$. The exact values lie within the hatched area. The latter uncertainty occurs since the number of test functions n is finite. We cannot expect that in the numerical results the lower bounds (obtained from the maximum principle) exactly agree with the upper bounds (obtained from the minimum principle). In addition, numerical inaccuracies are expected. To discriminate between these two sources, i.e., finite number of test functions and numerical errors, we have also developed complementary variational principles for the finite-dimensional matrix problems. For each eigenvalue computation they show the maximum numerical errors. In our calculations numerical errors lie within drawing accuracy. For small l, the growth rate γ_l increases proportionally to l. This agrees with the earlier calculations by Larsen[6] as well as Ablowitz and Segur.[6] For finite l, a maximum occurs and for even larger l, γ_l diminishes. The cutoff, however, lies outside the region of validity of our variational formulation. For fixed φ_0/h, the maximum growth rates increase with larger kh values. This is also shown in Fig. 3. Now, for the first time, we have results for the maximum growth rates which determine observability of the solitons. Taking, for example, $\varphi_0/h \simeq 2\times10^{-3}$ and $kh \simeq 15$ we obtain

$$\gamma_l \simeq 6\times10^{-4}\omega, \tag{19}$$

which for $h \simeq 10^3$ m yields $\gamma_l \simeq 2\times10^{-4}$ sec^{-1}. Thus the lifetime of such a soliton is approximately 1 h.

In Fig. 3, the asymptotic behavior of the maximum growth rate $\gamma_{\max}$ for large kh values can be approximated by the formula

$$\gamma_{\max}^2 \simeq 0.1 g^4\varphi_0^4\nu^2/\omega^4, \tag{20}$$

which, for $kh\to\infty$, approaches the Schrödinger

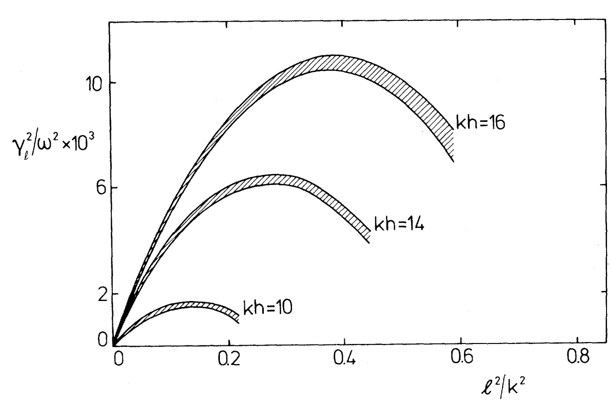

FIG. 2. Normalized growth rates vs transverse wave number for fixed $\varphi_0/h = 0.025$ and various parameters kh in the case of the Davey-Stewartson equations (1) and (2). The exact values lie within the hatched area.

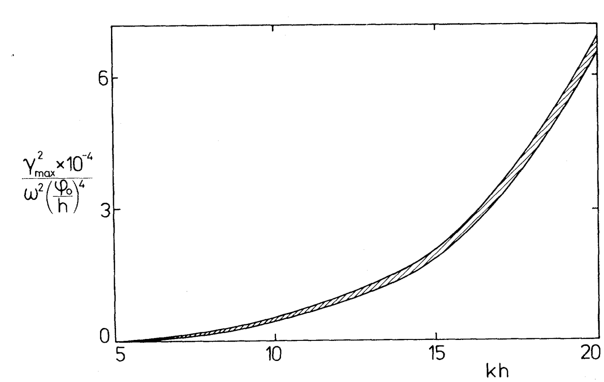

FIG. 3. Maximum growth rate $\gamma_{\max}$ vs kh for the Davey-Stewartson equations.

result

$$\gamma_{max}^2 \simeq 0.4(\varphi_0 k)^4 \omega^2, \tag{21}$$

since $\nu \to 2k^{7/2}g^{-1/2}$ for $kh \to \infty$. Thus we have excellent agreement with the deep-water limit.

Our calculations have further shown that modes with transverse wave number

$$l \simeq \tfrac{1}{2}\varphi_0 g \nu^{1/2}/\omega\mu^{1/2} \tag{22}$$

are most unstable.

In summary, we have presented the maximum growth rates for transverse instability of envelope water solitons in systems of finite depth. Our results show for the first time that although solitons are unstable, the decay rates can be so small that envelope solitons are observable for a long time. We have also depicted the wave-number dependence of the transverse growth rates. A clear prediction of the growth time and wave number of the most unstable mode was presented.

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[7]By introducing the standard length h and the standard time ω^{-1}, dimensionless quantities, e.g., $\bar{A} = A/h^2\omega$, can be used. Then the quantity $\bar{g} = g/h\omega^2$ will appear. From $\bar{g} \sim O(1)$, i.e., $\epsilon \ll \bar{g} \ll \epsilon^{-1}$, where $\epsilon = \varphi_0/h$, the region of validity of the Davey-Stewartson equations, i.e., $\epsilon^{1/2} \ll kh \ll \epsilon^{-1}$ can be deduced.

[8]As has been mentioned already by Larsen, Ref. 6, in the printing of the Davey-Stewartson paper, Ref. 3, a factor ω was missing in the expression for ν_1.

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