# Squeezed States and Sub-Poissonian Photon Statistics

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It is pointed out that, although squeezing and sub-Poissonian photon statistics need not go together, in the sense that an electromagnetic field may exhibit one but not the other, the method that is normally used to detect a squeezed state automatically generates sub-Poissonian photon statistics. However, when these considerations are applied to the fluorescence from a coherently driven atom, which exhibits both squeezing and sub-Poisson fluctuations, one finds that the statistics of the emitted photons show even larger departures from classical field theory than the squeezing.

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The nonclassical character of the so-called squeezed quantum states has recently been discussed in connection with the phenomenon of resonance fluorescence from an atom.[1] It was pointed out by Walls and Zoller[1] that, like states of the electromagnetic field that exhibit sub-Poissonian photon counting statistics, squeezed states have no classical analog, because their diagonal coherent-state representation cannot be nonnegative.[2,3] On the other hand, as the authors correctly point out, there is in general no direct connection between squeezing and sub-Poissonian statistics; states exist that exhibit the first but not the second, and vice versa. The phenomenon of atomic resonance fluorescence is distinguished in that the electromagnetic field exhibits both squeezing[1] and sub-Poissonian photon statistics[4-8] at the same time. In practice the squeezed state is normally identified by phase-sensitive interference with another optical field in a coherent state, followed by photoelectric detection of the resulting intensity fluctuations. I would like to draw attention to the fact that under these circumstances the squeezing always gives rise to sub-Poissonian photon statistics. On the other hand, when these conclusions are applied to the phenomenon of atomic resonance fluorescence, one finds that the departures from Poissonian photon statistics generated by such phase-sensitive interference are always smaller than those produced directly in the process of atomic emission. The general problem of detecting squeezed states in various ways has been treated theoretically in some detail.[9,10]

In the following we adopt a notation that is similar to that used in Ref. 1. Let $\hat{E}$ be a real electromagnetic field amplitude, with positive- and negative-frequency parts $\hat{E}^{(+)},\hat{E}^{(-)}$. Then we write[11]

$$\hat{E}_1=\hat{E}^{(+)}+\hat{E}^{(-)},\quad \hat{E}_2=-i(\hat{E}^{(+)}-\hat{E}^{(-)}),\tag{1}$$

for the two components of the field that are 90° out of phase. Although $\hat{E}$ could be the amplitude of the electric vector, it will be a little more convenient to define $\hat{E}$ in such a way that $\langle\hat{E}^{(-)}\hat{E}^{(+)}\rangle$ represents the average flux of photons of the electromagnetic field. If we write

$$[\hat{E}^{(+)},\hat{E}^{(-)}]=C,\tag{2}$$

for the commutator, which is a positive $c$ number, then the corresponding commutator of $\hat{E}_1$ and $\hat{E}_2$ is given by

$$[\hat{E}_1,\hat{E}_2]=2iC,\tag{3}$$

and the dispersions of $\hat{E}_1$ and $\hat{E}_2$ are

$$\begin{aligned}\langle(\Delta\hat{E}_1)^2\rangle&=C+\langle:(\Delta\hat{E}_1)^2:\rangle,\\ \langle(\Delta\hat{E}_2)^2\rangle&=C+\langle:(\Delta\hat{E}_2)^2:\rangle.\end{aligned}\tag{4}$$

As the squeezed state has been defined[1] by the requirement that either $\langle(\Delta\hat{E}_1)^2\rangle$ or $\langle(\Delta\hat{E}_2)^2\rangle$ be below the minimum-uncertainty product $C$, it follows immediately from Eq. (3) that either

$$\langle:(\Delta\hat{E}_1)^2:\rangle<0 \quad\text{or}\quad \langle:(\Delta\hat{E}_2)^2:\rangle<0\tag{5}$$

in a squeezed state. As was pointed out in Ref. 1, this cannot be achieved in any quantum state for which there exists a classical analog.

On the other hand, with the help of the commutation relations, the variance of the photon number operator $\hat{n}$ in any quantum state of the electromagnetic field can always be expressed in the form

$$\langle(\Delta\hat{n})^2\rangle=\langle\hat{n}\rangle+\langle:(\Delta\hat{n})^2:\rangle,\tag{6}$$

from which it follows that fluctuations less than Poissonian require $\langle:(\Delta\hat{n})^2:\rangle<0$, and this again has no classical counterpart.

In order to detect the in-phase or the out-of-phase components $\hat{E}_1$ and $\hat{E}_2$ of the field in the squeezed state described by some density operator $\hat{\rho}$, one would generally set up an interference experiment, in which the light from a laser centered on the same midfrequency is superimposed. We idealize this by taking the interfering light to be in a pure coherent state $|\{v\}\rangle$, with

$$|\{v\}\rangle = \hat{D}(\{v\})|\{0\}\rangle,$$

where $\{v\}$ is a multimode set of complex amplitudes, and $\hat{D}(\{v\})$ is the displacement operator.[3] Since $|\{v\}\rangle$ is the right eigenstate of $\hat{E}^{(+)}$ we shall write

$$\hat{E}^{(+)}|\{v\}\rangle = \mathcal{E}|\{v\}\rangle, \tag{7}$$

where $\mathcal{E}$ is some complex amplitude. Then the density operator of the resulting field has the form

$$\hat{D}(\{v\})\hat{\rho}\hat{D}^\dagger(\{v\}),$$

and the expectation value of any dynamical variable $\hat{O}$ is given by

$$\mathrm{Tr}[\hat{O}\hat{D}(\{v\})\hat{\rho}\hat{D}^\dagger(\{v\})] = \mathrm{Tr}[\hat{D}^\dagger(\{v\})\hat{O}\hat{D}(\{v\})\hat{\rho}] = \langle\hat{D}^\dagger(\{v\})\hat{O}\hat{D}(\{v\})\rangle, \tag{8}$$

where the symbol $\langle\cdots\rangle$ always denotes the quantum expectation in the original state $\hat{\rho}$. The field resulting from the interference can be shown to have the same dispersions in $\hat{E}_1$ and $\hat{E}_2$ as the original field, for we observe that

$$\langle\hat{D}^\dagger(\{v\})\hat{E}_1{}^2\hat{D}(\{v\})\rangle - \langle\hat{D}^\dagger(\{v\})\hat{E}_1\hat{D}(\{v\})\rangle^2$$
$$= \langle(\hat{E}^{(+)} + \mathcal{E} + \hat{E}^{(-)} + \mathcal{E}^*)^2\rangle - \langle(\hat{E}^{(+)} + \mathcal{E} + \hat{E}^{(-)} + \mathcal{E}^*)\rangle^2 = \langle\hat{E}_1{}^2\rangle - \langle\hat{E}_1\rangle^2, \tag{9}$$

and similarly for $\hat{E}_2$. In deriving this equation use has been made of the displacement property of the $\hat{D}(\{v\})$ operator,[3]

$$\hat{D}^\dagger(\{v\})\hat{E}^{(+)}\hat{D}(\{v\}) = \hat{E}^{(+)} + \mathcal{E}. \tag{10}$$

Let us now examine the photon statistics of the superposed field, which are different from the original. The average number of photons counted by a detector in some sufficiently short time interval $T$ can be expressed in the form

$$\langle n\rangle = \alpha T\langle\hat{D}^\dagger(\{v\})\hat{E}^{(-)}\hat{E}^{(+)}\hat{D}(\{v\})\rangle = \alpha T\langle(\hat{E}^{(-)} + \mathcal{E}^*)(\hat{E}^{(+)} + \mathcal{E})\rangle, \tag{11}$$

where $\alpha$ is a constant characterizing the collection efficiency and the quantum efficiency of the detector. Similarly one can express the difference between the variance and the mean as

$$\langle(\Delta n)^2\rangle - \langle n\rangle = (\alpha T)^2[\langle\hat{D}^\dagger(\{v\})\hat{E}^{(-)}\hat{E}^{(-)}\hat{E}^{(+)}\hat{E}^{(+)}\hat{D}(\{v\})\rangle - \langle\hat{D}^\dagger(\{v\})\hat{E}^{(-)}\hat{E}^{(+)}\hat{D}(\{v\})\rangle^2]$$
$$= (\alpha T)^2[\langle(\hat{E}^{(-)} + \mathcal{E}^*)^2(\hat{E}^{(+)} + \mathcal{E})^2\rangle - \langle(\hat{E}^{(-)} + \mathcal{E}^*)(\hat{E}^{(+)} + \mathcal{E})\rangle^2].$$

With the help of Eqs. (1) and (2) we may readily show that

$$\langle(\Delta n)^2\rangle - \langle n\rangle = (\alpha T)^2|\mathcal{E}|^2[\langle:(\Delta\hat{E}_1)^2:\rangle\cos^2\theta + \langle:(\Delta\hat{E}_2)^2:\rangle\sin^2\theta + \tfrac{1}{2}(\Delta\hat{E}_1\Delta\hat{E}_2 + \Delta\hat{E}_2\Delta\hat{E}_1)\sin 2\theta]$$
$$+ (\alpha T)^2[\langle\hat{E}^{(-)2}\hat{E}^{(+)2}\rangle - \langle\hat{E}^{(-)}\hat{E}^{(+)}\rangle^2 + 2\mathcal{E}\langle\Delta\hat{E}^{(-)}\Delta(\hat{E}^{(-)}\hat{E}^{(+)})\rangle + \mathrm{c.c.}], \tag{12}$$

after introducing the polar angle $\theta$ by writing $\mathcal{E} = |\mathcal{E}|\exp(i\theta)$.

If the coherent light beam is made sufficiently intense, the terms in $|\mathcal{E}|^2$ in Eq. (12) can be made to dominate over those in $\mathcal{E}$ and those without $\mathcal{E}$, which will henceforth be discarded. By changing the phase angle $\theta$ we can therefore examine the fluctuations of $\hat{E}_1$ and $\hat{E}_2$ in turn. Thus, we obtain

$$\langle(\Delta n)^2\rangle - \langle n\rangle \approx \begin{cases} (\alpha T)^2|\mathcal{E}|^2\langle:(\Delta\hat{E}_1)^2:\rangle, & \text{if } \theta = 0, \\ (\alpha T)^2|\mathcal{E}|^2\langle:(\Delta\hat{E}_2)^2:\rangle, & \text{if } \theta = \pi/2, \end{cases} \tag{13}$$

and it follows from Eq. (5) that if either the inphase or the out-of-phase component is squeezed, then the associated photon counting statistics are sub-Poissonian. Although this conclusion is already contained in Refs. 9 and 10, it is not easily extracted therefrom. Sometimes it is convenient to introduce the parameter $Q$ defined by

$$Q \equiv [\langle(\Delta n)^2\rangle - \langle n\rangle]/\langle n\rangle \tag{14}$$

as a measure of the departure from Poisson statistics.[6] It is clear from Eq. (11) that in a strong coherent field $\langle n\rangle \approx \alpha T|\mathcal{E}|^2$, so that we obtain

$$Q \approx \alpha T\langle:(\Delta\hat{E}_i)^2:\rangle, \quad i = 1, 2. \tag{15}$$

$Q$ is therefore always negative for one of the variables $\hat{E}_1, \hat{E}_2$ in the squeezed state when the phase angle $\theta$ is properly chosen.

A general remark concerning squeezed states in classical optics may be appropriate here. Classically, there are no restrictions on the fluctuations of the in-phase and out-of-phase components $E_1, E_2$ of the electromagnetic field amplitude, and either or both of the dispersions of $E_1, E_2$ can, in principle, be below $C$, although the two dispersions are equal if $E^{(+)}$ and $E^{(-)}$ are analytic signals. Such classical fields have occasionally been introduced in the discussion of the problem of gravitational wave detection.[12] Although the optical field could be described as being in a classical squeezed state, the photoelectric counting fluctuations produced by such a classical field will always be greater than Poissonian, both before and after interference with a coherent field. In this sense the classical squeezed state is always clearly distinguishable from its quantum mechanical counterpart.

Let us now apply these considerations to the problem of resonance fluorescence from a two-level atom. It may readily be shown that in the steady state, and for a certain phase of the field,[1,5]

$$\langle :(\Delta \hat{E}_i)^2: \rangle = 2\beta[\langle (\Delta \hat{\sigma}_i)^2 \rangle + \tfrac{1}{2}\langle \hat{\sigma}_3 \rangle], \quad i = 1, 2 \qquad (16)$$

in our units, where $\hat{\sigma}_i$ $(i = 1, 2, 3)$ is half the $i$th Pauli spin operator, and the parameter $2\beta$ is the Einstein $A$ coefficient for the excited atomic state. From expressions for $\langle \hat{\sigma}_i \rangle$ one finds that maximum squeezing in the steady state is obtained when either[1]

$$\langle :(\Delta \hat{E}_1)^2: \rangle = -\tfrac{1}{16}\beta \quad \text{and} \quad \langle :(\Delta \hat{E}_2)^2: \rangle = \tfrac{1}{8}\beta, \qquad (17)$$

or vice versa, and from Eq. (15) this gives a maximum negative $Q$ value of

$$Q = -\tfrac{1}{16}\alpha\beta T. \qquad (18)$$

Walls and Zoller have pointed out that $-Q$ can become twice as large in the transient regime of atomic fluorescence.[1]

This result may now be compared with the maximum departure from Poisson statistics achievable by counting the photons emitted by the driven atom directly. As Eq. (18) applies to a short-time counting situation, the comparison is meaningful only if we impose a similar restriction on the direct photon-counting experiment. From Eq. (11b) of Ref. 6 one easily finds by expanding in powers of $\beta T$ that, for $\beta T \ll 1$, $Q$ has the largest negative value of

$$Q = -\alpha\beta T. \qquad (19)$$

We therefore conclude that the problem of detecting the squeezing of the electromagnetic field radiated by a single coherently driven atom is at least an order of magnitude more difficult than the (already nontrivial) problem of observing the non-Poissonian photon statistics directly. In the sense of maximizing $-Q$, the photon emission from an atom is even more "nonclassical" than the squeezing.

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