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Exactly Solvable Model of a Physical System Exhibiting Universal Chaotic Behavior

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A model of a simple nonlinear physical system, the driven diode resonator comprised of an oscillator, resistor, inductor, and diode in series, is shown to reduce exactly to a one-dimensional, noninvertible map. With use of a model of the diode which includes the forward bias voltage, reverse recovery time, and junction capacitance, the response of the system is calculated exactly. The solution exhibits the period-doubling route to chaos with universal scaling.

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Period doubling and chaotic behavior were recently reported by Linsay[1] and by Testa, Pérez, and Jeffries[2] for the response of a driven anharmonic resonator consisting of a series circuit composed of a resistance, inductance, and a Varactor diode as shown in Fig. 1(a). This diode resonator was shown to follow a patterned route to chaos in good agreement with the universal behavior found in iterated, unimodal, one-dimensional maps.[3-6] Several recent experiments on a variety of nonlinear physical systems have revealed similar patterned chaotic behavior[1,2,7] and there is an active effort to develop an understanding of the dynamics of complex nonlinear physical systems by using much simpler models which are universally applicable to large classes of these systems.[4,6,8,9] It is important to have a clear understanding of the physical conditions present in the nonlinear system which lead to its approximate description by a simple model such as a one-dimensional map. It is the purpose of this paper to present a realistic physical model of the nonlinear diode resonator and to show that this model provides, for the first time, an exact description of the response in terms of a one-dimensional, noninvertible mapping function which is explicitly defined.

The previous work[1,2] on the diode resonator attributes the period doubling and chaotic behavior partially to the nonlinearity introduced by the voltage-dependent capacitance of the Varactor. However, Hunt[10] commented that another property of such diodes was responsible for the behavior; namely, the rather large reverse recovery time. We show here that *both* a finite forward bias voltage *and* a finite reverse recovery time are required if the diode resonator is to exhibit chaotic behavior. We totally neglect the changing capacitance of the Varactor and believe that it is unimportant with regard to the salient features of the response. We assume that the diode will be-

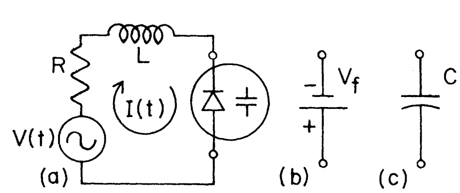

FIG. 1. (a) Driven Varactor-diode resonator circuit. (b) When the diode is conducting, it is replaced by an emf $= V_f$. (c) When the diode is off, it is replaced by capacitance C.

have as an ideal diode with the following added characteristics: (1) There is a finite forward bias voltage V_f. The diode will not conduct until the forward voltage drop reaches V_f and the voltage drop remains at V_f as long as the diode conducts. (2) When the voltage drop is less than V_f, the diode does not conduct but acts as a capacitor with *fixed* capacitance C. (3) When the current through the diode is driven through zero, the diode does not shut off immediately, but continues to conduct for a time equal to the reverse recovery time τ_r. While it is known that τ_r depends on various factors such as the magnitude of the forward current and the magnitude of the reverse voltage,[11] we choose a very simple functional relationship to describe τ_r:

$$\tau_r = \tau_m [1 - \exp(-|I_m|/I_c)], \tag{1}$$

where $|I_m|$ is the magnitude of the most recent maximum forward current, and τ_m and I_c are parameters which describe the characteristics of the particular diode used.

The response of the diode resonator can now be exactly calculated by splitting time into intervals when the diode is either conducting or off. Exact analytic solutions are easily obtained for each time interval and can be fitted together by the appropriate boundary conditions at the times when the diode is switching. Thus, we use alternately one and then the other of the circuits shown in Figs. 1(b) and 1(c).

Given the drive voltage $V(t) = V_0 \cos\omega t$, the general solution when the diode is conducting [Fig. 1(b)] is

$$I(t;A) = (V_0/Z_a)\cos(\omega t - \theta_a) + Ae^{-Rt/L} + V_f/R; \quad V_d(t) = -V_f, \tag{2}$$

where $Z_a^2 = R^2 + \omega^2 L^2$, $\theta_a = \arctan(\omega L/R)$, and A is a constant determined by boundary conditions. The general solution when the diode is not conducting [Fig. 1(c)] is

$$I(t;B,\Phi) = (V_0/Z_b)\cos(\omega t - \theta_b) + Be^{-2Rt/L}\cos(\omega_b t + \Phi), \tag{3}$$

$$V_D(t;B,\Phi) = V_0 \cos\omega t - I(t;B,\Phi)R - L\dot{I}(t;B,\Phi),$$

where $Z_b^2 = R^2 + (L/\omega)^2(\omega^2 - \omega_0^2)^2$, $\theta_b = \arctan[L(\omega^2 - \omega_0^2)/R\omega]$, $\omega_0^2 = 1/LC$, $\dot{I} = dI/dt$, $\omega_b^2 = \omega_0^2 - (R/2L)^2$, and B and Φ are constants to be determined by boundary conditions.

Time is divided into cycles as shown in Fig. 2. The solutions are determined during the nth cycle by the set of constants (A_n, B_n, Φ_n) together with the times $t_1(n)$, $t_2(n)$, and $t_3(n)$. The initial time of the nth cycle, $t_1(n)$, is the beginning of a time interval when the diode is conducting. The boundary condition at $t_1(n)$ requires the current and voltage across the diode to be continuous and determines the constant A_n. At $t_2(n)$ the current crosses zero while the diode is conducting. An ideal diode would stop conducting at $t_2(n)$; however, the diode continues to conduct for a time interval, τ_r, given by Eq. (1) using the value of $|I_m|$ found during the time interval $t_1(n) < t < t_2(n)$. Time $t_3(n) = t_2(n) + \tau_r(n)$. If the current at $t_3(n)$ is passing through the diode in the reverse direction, the diode stops conducting at $t_3(n)$ and the circuit switches to that of Fig. 1(c). In this case the boundary conditions at $t_3(n)$ determine B_n and Φ_n. The current and voltage are thus determined by Eq. (3) until $V_d(t)$ comes back to $-V_f$, where we reach $t_1(n+1)$ and the cycle starts over. If the current at $t_3(n)$ is passing through the diode in the forward direction, then the diode does not

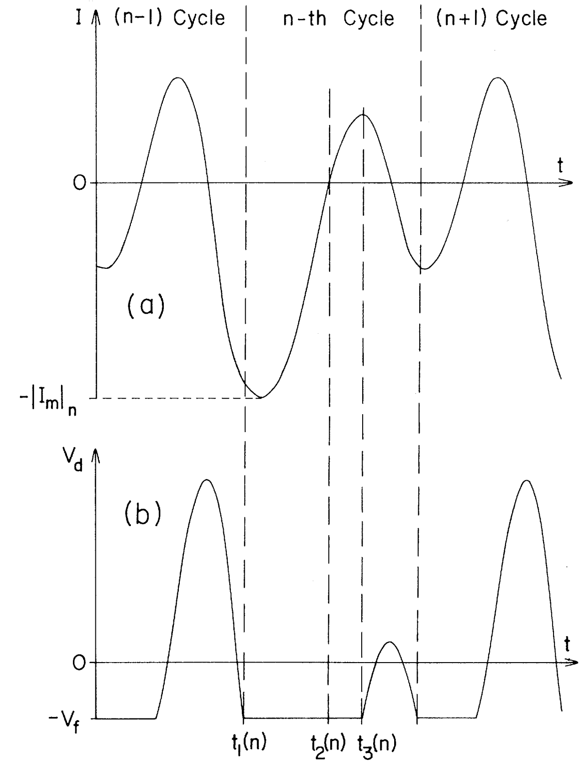

FIG. 2. Typical response of the driven diode resonator calculated with use of the model described in the text. The response for the case shown has a period twice that of the drive period. The times $t_i(n)$ which determine the nth cycle are described in the text. (a) Current vs time for three consecutive cycles. (b) Voltage across the diode vs time.

stop conducting at $t_3(n)$, but rather the next cycle is started with $t_1(n+1)=t_3(n)$.

Given an initial value of A_1, we can use the above step-by-step process to determine $I(t)$. In fact, it is enough to be given $|I_m|$, the maximum forward current, during one cycle to determine $I(t)$ uniquely at all subsequent times. The value of $|I_m|$ together with the condition that $I(t)$ is minimum at that point establishes both the time t' at which the minimum occurs and A_1. The conditions are

$$I(t';A_1)=-|I_m|; \quad \dot{I}(t';A_1)=0; \quad \ddot{I}(t';A_1)>0; \tag{4}$$

where $I(t)$ is given by Eq. (2). The first two conditions give the following equation for t':

$$V(t)=V_0\cos\omega t'=-|I_m|R-V_f. \tag{5}$$

[This condition is also evident by inspection of Fig. 1(b).] The domain for t' may be limited to the interval $0<t'<2\pi/\omega$ without loss of physical generality. Within this interval there are two distinct solutions to Eq. (5). However, the third condition of Eqs. (4) leads to the condition $-\sin\omega t'>0$ and only one of the solutions survives. Having determined t', we use Eqs. (2) and (4) to determine A_1:

$$A_1=-(V_0L\omega/RZ_a)\sin(\omega t'-\theta_a)\exp(Rt'/L).$$

This leads to the unique determination of I for all subsequent times.

Thus, the proposed model for the diode in a series diode resonator circuit leads directly to an exact one-dimensional mapping function for this nonlinear physical system. Successive values of $|I_m|$ may be expressed as iterations of a one-parameter family of one-dimensional maps of the form

$$|I_m|_n \to |I_m|_{n+1}=F(|I_m|_n;V_0),$$

where $|I_m|_n$ is the maximum forward current through the diode during the nth cycle, V_0 is the magnitude of the drive voltage, and the mapping function, F, is given by the step-by-step procedure described above. The function F contains the circuit parameters R, L, C, τ_m, and I_c in addition to ω and V_0. However, in a given experiment[1,2] the circuit parameters and drive frequency are held fixed and the response is studied as a function of the drive voltage V_0. It is clear that this system will exhibit behavior consistent with the properties of iterated one-dimensional maps as described by Feigenbaum[5] and others.[6]

In order for a one-dimensional map to exhibit chaotic behavior, it must be noninvertible.[8] It should be noted that the mapping function described above is noninvertible, i.e., given $|I_m|_{n+1}$ we cannot always determine $|I_m|_n$ uniquely. Time can be reversed in the solutions given by Eqs. (2) and (3), but τ_r cannot be determined directly unless $|I_m|$ at the *earlier* time is known. Inverting the process requires applying self-consistency between τ_r and the earlier $|I_m|$ which results. Computer calculations of F discussed below show that the map is noninvertible and that there may be two solutions to this self-consistency problem.

We have performed computer experiments mirroring the measurements previously reported.[2] The computer calculations were done by starting the resonator at $t=t_1=\pi/2\omega$ with the diode conducting and $I(t_1)=0$ (which determines A_1) and then following the step-by-step procedure described above using sixteen-place precision. Typical circuit parameters were as follows: $\omega\cong\omega_0$; $Q\equiv L\omega_0/R$ between 10 and 50; $\tau_m\cong 2\pi/\omega_0\equiv T_0$ and I_c such that $0<\tau_r<0.4T_0$. For $\lambda\equiv V_0/V_f=2.0$, $I(t)$ was found to converge quickly to a periodic function with the period equal to that of the drive, $2\pi/\omega$. As the parameter λ was increased, we observed the period-doubling bifurcation route to chaos expected for a unimodal one-dimensional map.[6] For λ in the periodic region the solution became periodic to within one part in 10^9 after a few hundred cycles as long as λ was not too close to a bifurcation point. At birfurcation point, λ_n, the period doubles to 2^n times that of the drive, where $n=1, 2, \ldots$. A careful study of the values of λ_n was performed and the scaling parameters $\delta_n\equiv(\lambda_n-\lambda_{n-1})/(\lambda_{n+1}-\lambda_n)$ were calculated. For the circuit parameters $\omega=0.9\omega_0$, $Q=50$, $\tau_m=2\pi/\omega(0.9)^2$, and $I_cR/V_f=1.0$, successive bifurcations were observed at $\lambda_1=2.40(1)$, $\lambda_2=5.070(5)$, $\lambda_3=6.184(1)$, $\lambda_4=6.473(1)$, $\lambda_5=6.5351(2)$, and $\lambda_6=6.548\,80(5)$. The estimated errors in the last displayed significant figures are shown in parentheses. We obtain $\delta_2=2.40(2)$, $\delta_3=3.85(5)$, $\delta_4=4.66(20)$, and $\delta_5=4.53(15)$. These are consistent with the universal limiting value[5] $\delta_\infty=4.669$. For $\lambda>\lambda_\infty$ the solution appeared to be chaotic except for windows in λ where solutions with period 3, 6, and 5 were observed, similar to the behavior reported by Testa, Pérez, and Jeffries.[2]

A numerical calculation of the mapping function F may be obtained by plotting $|I_m|_{n+1}$ vs $|I_m|_n$ in the chaotic region. Figure 3 shows the mapping function obtained for $\lambda=18$. The map for an orbit

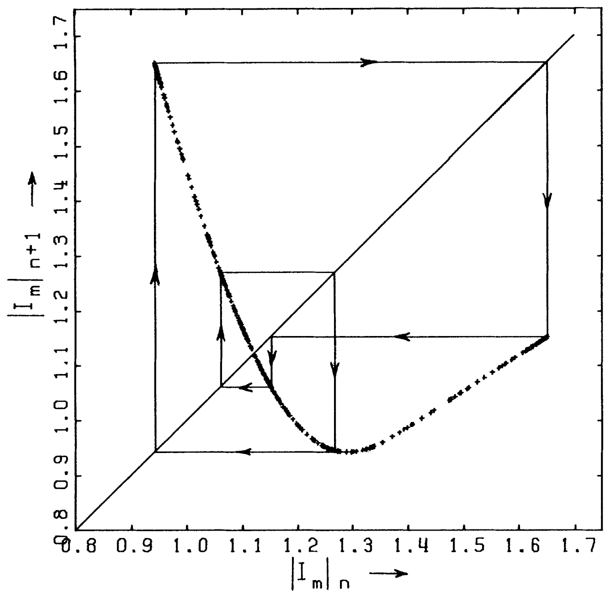

FIG. 3. The one-dimensional mapping function $|I_m|_{n+1}$ vs $|I_m|_n$ (in units of V_f/R) for the 744th through the 999th cycle for $V_0/V_f = 18$ in the chaotic region. The circuit parameters in this case were $\omega = \omega_0$, $Q = 15$, $\tau_m = 2\pi/\omega$, and $I_cR/V_f = 2.0$.

which nearly has period 5 is also indicated in Fig. 3. A transformation of the type $X_n = \text{const} - |I_m|_n$ will turn the mapping function upside down so that it has the rounded maximum required of a smooth unimodal mapping function.[6] The mapping function shown in Fig. 3 is noninvertible since certain values of $|I_m|_{n+1}$ may be reached from two different values of $|I_m|_n$.

The stability of periodic orbits depends on the shape of the mapping function which, in turn, depends on the circuit parameters and V_0. We have found that if *either* the forward bias voltage, V_f, *or* the reverse recovery time, τ_r, of the diode is set equal to zero, no bifurcations occur and the period-one solution is always stable. We are conducting a detailed study of the shape of the mapping function F and its dependence on the circuit and drive parameters.

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