Transverse Electromagnetic Waves with Finite Energy, Action, and $\int \vec{E}\cdot\vec{B}\, d^4x$

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Transverse electromagnetic waves possessing finite energy, action, and $\int \vec{E}\cdot\vec{B}\, d^4x$ are obtained in 3 + 1 dimensions as a solution of the source-free Maxwell's equation in vacuum.

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In the last few years finite-energy solutions of classical Yang-Mills theory have attracted a great deal of attention.[1] Most of these solutions are stable, at least at the classical level, because of the existence of some conserved (non-Noether) charge. For example, the instanton and the meron solutions have nonzero pseudoscalar charge q defined by

$$q = -\int d^4x\, \vec{E}_a\cdot\vec{B}_a. \tag{1}$$

On the other hand, not much work seems to have been done regarding the finite-energy, stable solutions of classical electrodynamics. One of the possible reasons for this may be the notion that for transverse electromagnetic waves the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are perpendicular to each other. However, very recently, Chu and Ohkawa[2] have shown that one can have a class of transverse electromagnetic waves with $\vec{E}\parallel\vec{B}$. Unfortunately the field energy for their solution diverges. Besides, not only q but even the time average of $\vec{E}\cdot\vec{B}$ is zero in the case of their solution.

The purpose of this Letter is to obtain transverse electromagnetic waves in 3+1 dimensions possessing finite field energy, finite action, and finite q as a solution of the source-free Maxwell's equation in vacuum.

We start with the *Ansatz* $(c=1)$

$$\vec{A}(\vec{x},t) = \sum_{\vec{K}} \frac{C_K}{(2\pi)^{3/2}}(\vec{a}\sin\vec{K}\cdot\vec{x} + \vec{b}\cos\vec{K}\cdot\vec{x})\cos(Kt+\alpha), \tag{2}$$

where we have chosen

$$\vec{K} = (K/\sqrt{3})(1,1,1) \tag{3}$$

and C_K is a momentum-space weight factor whose exact form will be specified later. Further a_i, b_i, and α are dimensionless constants. We shall work in the Coulomb gauge, i.e., $A_0=0$, $\nabla\cdot\vec{A}=0$ which requires that

$$\textstyle\sum_i a_i = 0, \quad \sum_i b_i = 0. \tag{4}$$

The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be easily calculated and are found to be

$$\vec{E}(\vec{x},t) = -\frac{\partial\vec{A}(\vec{x},t)}{\partial t} = \sum_{\vec{K}}\vec{E}_K(\vec{x},t) = \sum_{\vec{K}}\frac{KC_K}{(2\pi)^{3/2}}(\vec{a}\sin\vec{K}\cdot\vec{x} + \vec{b}\cos\vec{K}\cdot\vec{x})\sin(Kt+\alpha), \tag{5}$$

$$B_1 = (\nabla\times\vec{A})_1 = \sum_{\vec{K}}\frac{KC_K}{(2\pi)^{3/2}}\left(\frac{a_3-a_2}{\sqrt{3}}\cos\vec{K}\cdot\vec{x} + \frac{b_2-b_3}{\sqrt{3}}\sin\vec{K}\cdot\vec{x}\right)\cos(Kt+\alpha). \tag{6}$$

(B_2 and B_3 can be similarly written down.) We now choose

$$\sqrt{3}\,b_1 = a_3 - a_2, \tag{7a}$$

$$\sqrt{3}\,a_1 = b_2 - b_3, \tag{7b}$$

and similar cyclic relations for b_2, b_3 and a_2, a_3, which ensure Eq. (4). With this choice Eq. (6) yields

$$\vec{B}(\vec{x},t) = \sum_{\vec{K}}\vec{B}_K(x,t) = \sum_{\vec{K}}\frac{KC_K}{(2\pi)^{3/2}}(\vec{a}\sin\vec{K}\cdot\vec{x} + \vec{b}\cos\vec{K}\cdot\vec{x})\cos(Kt+\alpha). \tag{8}$$

Inspection of Eqs. (5) and (8) reveals that $\vec{E} \parallel \vec{B}$ only if

$$\vec{a}\times\vec{b}=0, \quad (9)$$

which is satisfied by our choice of a and b. Note that Eq. (4) ensures

$$\vec{K}\cdot\vec{E}_K(\vec{x},t)=0, \quad \vec{K}\cdot\vec{B}_K(\vec{x},t)=0. \quad (10)$$

It is now straightforward to check that the $\vec{E}$ and $\vec{B}$ as given by Eqs. (5) and (8) satisfy Maxwell's equations

$$\nabla\cdot\vec{E}=0, \quad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \quad \nabla\cdot\vec{B}=0, \quad \nabla\times\vec{B}=\frac{\partial\vec{E}}{\partial t}. \quad (11)$$

Thus, unlike the conventional notion, we have obtained transverse electromagnetic waves in which $\vec{E}\parallel\vec{B}$. Hence for our solution, the field momentum $\vec{E}\times\vec{B}$ is zero. It is also not difficult to calculate the field energy ϵ, action S, and pseudoscalar charge q for our solution. One can show that

$$\epsilon=\tfrac{1}{2}\int d^3x\,(\vec{E}^2+\vec{B}^2)=a^2\int_0^\infty dK\,K^4C_K^{\,2}, \quad (12a)$$

$$q=-\int d^4x\,\vec{E}\cdot\vec{B}=a^2\sin 2\alpha\int_0^\infty dK\,K^4C_K^{\,2}\int_{-\infty}^{\infty}dt\,\cos 2Kt=\tfrac{1}{2}\pi a^2\sin 2\alpha\,(K^4C_K^{\,2})|_{K=0}, \quad (12b)$$

$$s=\tfrac{1}{2}\int d^4x\,(\vec{E}^2-\vec{B}^2)=-a^2\cos 2\alpha\int_0^\infty dK\,K^4C_K^{\,2}\int_{-\infty}^{\infty}dt\,\cos 2Kt=-\tfrac{1}{2}\pi a^2\cos 2\alpha\,(K^4C_K^{\,2})|_{K=0}. \quad (12c)$$

In the above we have taken $a_1=a_2=a/\sqrt{2}$ without any loss of generality.

From Eqs. (12b) and (12c) it is clear that in order to obtain nonzero q and s one must choose C_K of the type

$$\lim_{K\to 0} C_K=g(K)/K^2, \quad g(0)=1, \quad (13)$$

while in order that ϵ be finite it follows from Eq. (12a) that as $K\to\infty$, C_K must die off faster than $K^{-1/2}$.

As an illustration, we choose the following C_K which satisfies both of these requirements.

$$K^2C_K=e^{-K\lambda}, \quad (14)$$

where λ is an arbitrary constant with dimension of length. The appearance of λ is related to the fact that Maxwell's equations are scale invariant. On using this C_K in Eqs. (12a) to (12c) we get

$$\epsilon=a^2/2\lambda, \quad (15a)$$

$$q=\tfrac{1}{2}\pi a^2\sin 2\alpha, \quad (15b)$$

$$s=-\tfrac{1}{2}\pi a^2\cos 2\alpha. \quad (15c)$$

By choosing the phase α appropriately one can obtain solutions having $q>s$ or $q<s$.

One might wonder if our choice for C_K gives nonsingular $A_i(x)$. Using the C_K as given by Eq. (14) in Eq. (2) and performing k integration we find that

$$\vec{A}(\vec{x},t)=\frac{\vec{a}}{(2\pi)^{3/2}}\left(\frac{x(\lambda^2+x^2-t^2)\cos\alpha-2\lambda xt\sin\alpha}{[\lambda^2+(x-t)^2][\lambda^2+(x+t)^2]}\right)+\frac{\vec{b}}{(2\pi)^{3/2}}\left(\frac{\lambda(\lambda^2+x^2+t^2)\cos\alpha-t(\lambda^2-x^2+t^2)\sin\alpha}{[\lambda^2+(x-t)^2][\lambda^2+(x+t)^2]}\right), \quad (16)$$

where $\sqrt{3}x=(x_1+x_2+x_3)$. This is clearly nonsingular.

Since the solution obtained above is characterized by nonzero values of q and s and since q and s are gauge and Lorentz-invariant quantities,[3] hence the solution should be stable against decay to solutions with $q=0$, $s=0$. Whether quantum correction will destroy this solution or not has to be seen.

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[1]See, for example, the exhaustive review by A. Actor, Rev. Mod. Phys. 51, 461 (1979), and reference therein.
[2]C. Chu and T. Ohkawa, Phys. Rev. Lett. 48, 837 (1982).
[3]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., Chap. 12.