Spectrum of the Schrödinger Equation on a Self-Similar Structure

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The spectrum of the Schrödinger equation, with magnetic field, on a model self-similar structure is considered. Nesting properties are formulated. Low-field behavior of the spectrum edge (of interest for superconductive diamagnetism) is discussed. Comparison between self-similar structures and regular lattices is emphasized.

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Consider the triangular structure, built according to the construction depicted in Fig. 1. At stage 0, it is an equilateral triangle made of wires. Iteration from stage n to $n+1$ implies the juxtaposition of three stage-n structures. This is a simple self-similar structure, a so-called Sierpinski gasket, the fractal dimensionality[1] of which is $\bar{d}=\ln 3/\ln 2$, and it has been contemplated as a model for a percolation backbone.[2-4] We are interested in solving the Schrödinger equation (with suitable boundary conditions at the nodes) on such a network and in finding its spectrum (Landau levels) in the presence of a magnetic field. Since this is equivalent to solving the linearized Ginzburg-Landau equation, several diamagnetic superconducting properties of such a network are governed by the highest eigenvalue.[5]

One question is: How does the spectrum of a self-similar structure differ from the spectrum of a regular lattice structure?

The Schrödinger equation in the presence of a magnetic field is traditionally written as

$$[i\nabla+(2\pi/\Phi_0)\vec{A}]^2\ \psi=E\psi, \tag{1}$$

where $\vec{A}$ is the vector potential associated with a magnetic field H, normal to the plane of the gasket, and Φ_0 is the flux quantum. Projection of the vectors along the wire directions is meant.[5] The boundary conditions at the nodes, for the wave function ψ and its gradients, are the natural generalizations of Kirchhoff relations, ensuring current conservation.[5,6]

A few preliminary words about notational choice are in order. In the context of superconductivity, the eigenvalue E on the right-hand side of Eq. (1) represents ξ^{-2}, where ξ is the superconducting coherence length.[5] In the context of Landau levels, in a tight-binding formulation,[7] the useful quantity is the (dimensionless) energy ϵ defined by

$$E=l^{-2}(\arccos\epsilon/4)^2, \tag{2}$$

where l is the elementary length (edge length of the stage-0 triangle). The spectrum of ϵ is then confined to the interval $[-4,+4]$, for the gasket as well as for the square lattice.[7] For ease of comparison, and also because symmetry properties (which will become apparent below) make it preferable, we choose to discuss the spectrum in terms of the variable ϵ.

At stage 0, the spectrum is trivially obtained as

$$4z^3-3z=\cos(2\pi\Phi/\Phi_0), \tag{3}$$

where $\epsilon=4z$ and Φ is the magnetic flux through the elementary triangle $[\Phi=(\sqrt{3}/4)l^2H]$. In Fig.

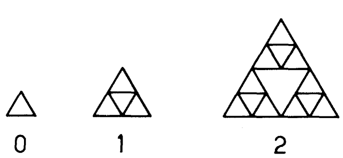

FIG. 1. Three first stages of the construction of an infinite triangular Sierpinski gasket. The linear size is increased by a factor of 2 at each iteration, while the total wire length increases by a factor of 3.

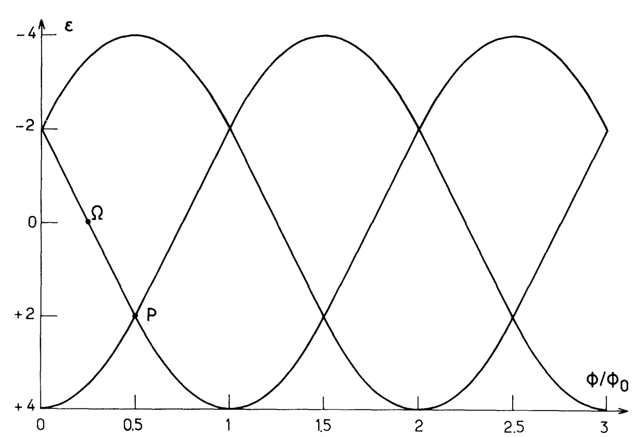

FIG. 2. Spectrum at state 0. The eigenvalues ϵ are plotted as functions of the ratio Φ/Φ_0, where Φ is the magnetic flux and Φ_0 the flux quantum. Note the symmetries of the spectrum (discussed in text) and in particular the symmetry with respect to the point Ω.

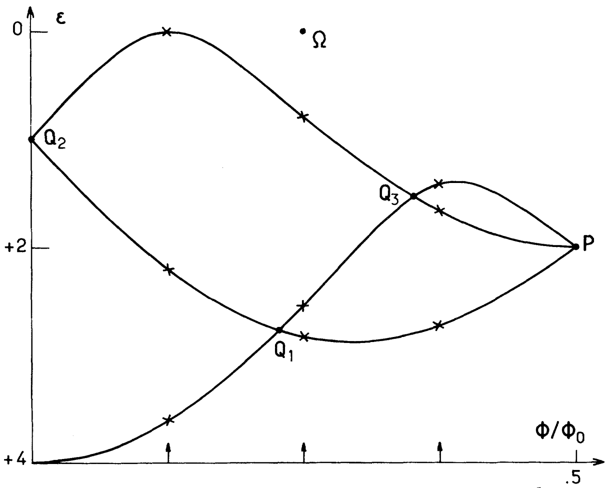

FIG. 3. Spectrum at stage 1. Point P is a triple point (crossing of three curves). Points Q_1, Q_2, and Q_3 are double points. The vertical arrows on the horizontal axis correspond to abscissas $\Phi/\Phi_0=\frac{1}{8},\frac{2}{8},\frac{3}{8}$. The corresponding eigenvalues are marked with crosses.

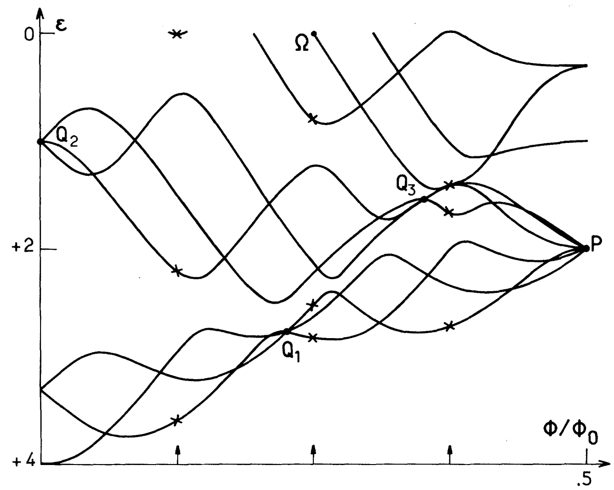

FIG. 4. Spectrum at stage 2. Point P is a point of degeneracy 6. Points Q_1, Q_2, and Q_3 are triple points. The vertical arrows and the crossed points are the same as in Fig. 3.

2, energy ϵ is plotted versus Φ/Φ_0, showing a well-known set of curves, exhibiting manifestly the flux periodicity imposed by the gauge invariance of Eq. (1).

At stage 1, the spectrum can still be obtained analytically. In Fig. 3, we have plotted only a part of the spectrum: $0\le\epsilon\le4$, $0\le\Phi/\Phi_0\le\frac{1}{2}$. The reason is that the spectrum has a symmetry point Ω ($\Phi/\Phi_0=\frac{1}{4}$, $\epsilon=0$), as visible in Fig. 2. The whole spectrum can be reconstituted by using the flux periodicity $\epsilon(\Phi/\Phi_0+k)=\epsilon(\Phi/\Phi_0)$, k integer, the symmetry of the spectrum around $\Phi/\Phi_0=\frac{1}{2}$, and the inversion symmetry around Ω. Accordingly, we may restrict our attention to the interval $0\le\Phi/\Phi_0\le\frac{1}{2}$, $0\le\epsilon\le4$.

Then it is observed that the lowest curve of the spectrum which was one arc at stage 0 is made of two arcs, joining at point Q_1, at stage 1. Besides, one notices that point P ($\Phi/\Phi_0=\frac{1}{2}$, $\epsilon=0$), which was a double point at stage 0, becomes a triple point at stage 1.

At stage 2, the spectrum can still be obtained without too much effort and it is shown in Fig. 4. As can be seen, point P becomes the crossing point of six energy levels, whereas point Q_1 becomes the crossing point of three branches. The lowest curve (highest eigenvalue) is now made of six arcs, with four new crossing points (besides Q_1).

In order to calculate the spectrum beyond stage 3, it is useful to derive recursion relations. Such relations are obtained,[3] by scaling down the problem of order n to that of order $n=0$, via a simple renormalization of parameters. Define $\Phi^{(n)}$ (respectively, $\tilde{\Phi}^{(n)}$) as the renormalized flux through a corner (respectively, central) triangle, and denote by z_n the renormalized value of z. The recursion equations, for which full credit should go to Alexander[3] who derived them first, are reproduced here in order to make this presentation self-contained:

$$\Phi^{(0)}=\tilde{\Phi}^{(0)}=\Phi, \tag{4a}$$

$$z_0=z, \tag{4b}$$

$$z_{n+1}=(z_n\Delta_n-a_n)/|b_n|, \tag{5}$$

$$\Phi^{(n+1)}=3\Phi^{(n)}+\tilde{\Phi}^{(n)}-3\frac{\Phi_0}{2\pi}\theta_n, \tag{6a}$$

$$\tilde{\Phi}^{(n+1)}+\Phi^{(n+1)}=3(\tilde{\Phi}^{(n)}+\Phi^{(n)})+2(4^n)\Phi \tag{6b}$$

where a_n, b_n, θ_n, and Δ_n are defined by

$$\Delta_n=64z_n^3-12z_n-2\cos(2\pi\tilde{\Phi}^{(n)}/\Phi_0), \tag{7a}$$

$$a_n=16z_n^2-1+4z_n\cos(2\pi\Phi^{(n)}/\Phi_0)+\cos[2\pi(\Phi^{(n)}+\tilde{\Phi}^{(n)})/\Phi_0], \tag{7b}$$

$$b_n=|b_n|\exp(-i\theta_n)=16z_n^2-1+4z_n\left[2\exp\left(-i\frac{2\pi\Phi^{(n)}}{\Phi_0}\right)+\exp\left(-i\frac{2\pi(2\Phi^{(n)}+\tilde{\Phi}^{(n)})}{\Phi_0}\right)\right]$$
$$+\left[\exp\left(-2i\frac{2\pi\Phi^{(n)}}{\Phi_0}\right)+2\exp\left(-i\frac{2\pi(\Phi^{(n)}+\tilde{\Phi}^{(n)})}{\Phi_0}\right)\right]. \tag{7c}$$

This set of nonlinear recursion equations can be viewed as a map $(z_n, \Phi^{(n)}) \to (z_{n+1}, \Phi^{(n+1)})$, because $\tilde{\Phi}^{(n)}$ can be eliminated via the following conservation equation:

$$\tilde{\Phi}^{(n)} + \Phi^{(n)} = 2(4^n)\Phi \tag{8}$$

which is a consequence of Eq. (6b).

A finite gasket, at stage n, can then be renormalized to a single triangle, by repeated use of the above equations. In particular, the eigenvalue equation of the gasket, at stage n, is

$$4z_n^3 - 3z_n = \cos(2\pi\Phi^{(n)}/\Phi_0), \tag{9}$$

which is the generalization of Eq. (3).

We are now in a position to discuss various properties of the spectrum.

(i) *Miscellaneous properties.*—It is possible to prove rigorously the symmetry properties of the spectrum, which were mentioned above and which allow attention to be restricted to the interval $0 \leq \epsilon \leq 4$, $0 \leq \Phi/\Phi_0 \leq \frac{1}{2}$. The inversion symmetry around point Ω is related to the triangular (frustrating) nature of the elementary cells. (For the square lattice, there is a symmetry between positive and negative ϵ.) Note that Ω belongs to the spectrum, for n even.

To the highest eigenvalue in zero field, $\epsilon = 4$, $\Phi/\Phi_0 = 0$, there corresponds an eigenfunction which is uniformly extended over the gasket. A uniformly extended eigenfunction is also obtained for $\epsilon = -4$, $\Phi/\Phi_0 = \frac{1}{2}$, as a consequence of the symmetry around Ω. The lowest eigenvalue in zero field, $\epsilon = -2$, $\Phi/\Phi_0 = 0$, is highly degenerate (like point P) as a result of the frustration effect.

(ii) *Nesting properties.*—Properties similar to the previous ones (i) can be found on regular lattice structures (e.g., triangular lattice[8]). We consider now properties which are intimately related to the self-similar nature of the gasket.

Nesting property I: If, for some value of the field, an eigenvalue is degenerate, at stage n, it will remain part of the spectrum, at all higher stages, with increasing degeneracy.

Consider, for instance, point P: $\epsilon = +2$, $\Phi/\Phi_0 = \frac{1}{2}$. At stage 0, it belongs to the spectrum (Fig. 2) with degeneracy 2 (crossing point of two branches); at stage 1, with degeneracy 3; at stage 2, with degeneracy 6. It can be shown[9] that its degeneracy is $\frac{3}{2}(1+3^{n-1})$ at stage n. As a second example, consider point Q_1 (Fig. 3), which is doubly degenerate at stage 1. At the next stage (Fig. 4), its degeneracy is 3.

This property is related to the Z_3 symmetry of the gasket geometry. The increase of the degeneracy at other crossing points seems to follow the same pattern as for point P. Though we have no formal proof yet for nesting property I, in its generality, we have accumulated empirical evidence, from observation of the first stages, and found no counterexample.

Nesting property II: The eigenvalues which, at stage n, correspond to equally spaced field values given by $\Phi/\Phi_0 = k/2(4^n)$, where k is an integer running from 0 to 4^n, will remain part of the spectrum at all higher stages, for the same field values. Moreover, the tangents, at these points, to the eigenvalue curves keep the same slope, at higher stages.

Consider, for example, the spectrum at stage 1, which is drawn in Fig. 3. The abscissas $\Phi/\Phi_0 = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}$ have been marked with a vertical arrow and the corresponding values are marked with a cross. These crossed points are still part of the spectrum at stage 2, as visible in Fig. 4.

We have been able to prove nesting property II. Our proof is based on straightforward, but cumbersome, algebra, using the recursion equations (4)–(6). For instance, the first statement derives from the following proposition: If, for a given stage n, the point $[\Phi/\Phi_0 = k/2(4^n), \epsilon]$ belongs to the spectrum, i.e.,

$$4z_n^3 - 3z_n = \cos(2\pi\Phi^{(n)}/\Phi_0),$$

then it follows that

$$4z_{n+1}^3 - 3z_{n+1} = \cos(2\pi\Phi^{(n+1)}/\Phi_0), \tag{10}$$

which implies that this point is also part of the spectrum at stage $n+1$.

Nesting properties I and II, taken together, suggest to what extend the spectrum at any stage is constrained by the spectrum of the previous stage. Comparison of the conspicuous nesting properties of translation-invariant lattices[7,8] and of dilatation-invariant structures appears as a physically attractive goal, of important potential significance.[10]

As a final remark, we give further consideration to the behavior of the spectrum edge (highest eigenvalue) in low field. The neighborhood of the point $\epsilon = 4$, $\Phi/\Phi_0 = 0$, which is a fixed point of the recursion equations, has been studied previously,[3,4] in some detail, and the following result established:

$$\epsilon \sim 4 - \tfrac{2}{99}[1 + 10(\tfrac{16}{5})^n](2\pi\Phi/\Phi_0)^2. \tag{11}$$

This implies that, for large size $L = 2^n l$,

$$\Delta\epsilon = 4 - \epsilon \sim H^2 L^{2-\bar{\delta}} \tag{12}$$

with $\bar{\delta} = \ln 5/\ln 2 - 2$. This same exponent $\bar{\delta}$ governs the behavior of the conductivity[2] and of the diffusion[11] on the gasket, in zero field. Such finite-size effects are due to the sensitivity of eigenfunctions to boundaries and they occur for $\Delta\epsilon$ small enough. A crossover is expected to take place around

$$\Delta\epsilon \sim L^{-2} \qquad (13)$$

toward an asymptotic (L independent) behavior, which is derivable from Eqs. (12) and (13) as

$$\Delta\epsilon \sim H^{4/(4-\bar{\delta})}. \qquad (14)$$

In fact, the edge of the spectrum is a curve which is the nonanalytic limit of a proliferating number of arcs (extrapolate from Figs. 2–4). The power law (14) describes its asymptotic low-field behavior. This is to be compared with the standard linear regime

$$\Delta\epsilon \sim H \qquad (15)$$

which is found on regular lattices[3,7,8,12] (see particularly Fig. 1 of Ref. 7). Thus the anomalous exponent $\bar{\delta}$, which is related to dilatation invariance, vanishes trivially for a regular lattice. However, it is presumably nonzero for a percolating cluster at threshold.[2,11] As a consequence, many physical properties around a percolation threshold, including diamagnetic superconducting properties, can be treated along the lines of the preceding scaling analysis.

This is another justification for studying gasket structures which, at first sight, may look somewhat artificial. Though the precise value of exponent $\bar{\delta}$ on the Sierpinski gasket is probably not pertinent for percolation, many qualitative aspects of the spectrum, with its nesting and scaling properties, bid fair to have physical relevance.

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