Recurrence Phenomena in Quantum Dynamics

In a Letter[1] with the same title as above, Hogg and Huberman considered quantum systems having time-periodic Hamiltonians with discrete quasienergy spectra. They showed that the evolution of these systems is almost-periodic (the initial state reappears infinitely often in the course of time). This phenomenon is more troublesome than the classical Poincaré recurrence. In the classical case, neighboring points in phase space may have vastly different recurrence times. On the other hand, it is possible to construct dense sets of neighboring quantum states with the *same* recurrence time. Therefore the coarse-grained entropy of such quantum ensembles does not increase to its equilibrium value, but is an almost-periodic function of time.[2]

In view of the far-reaching implications of this result, it is worthwhile to evaluate a typical recurrence time for such a quantum system. Let us take for simplicity a time-independent Hamiltonian H. Its eigenfunctions u_m satisfy $Hu_m = E_m u_m$, with incommensurable energy levels $E_m = 2\pi\hbar\nu_m$. The initial wave function can be expanded as $\Psi = \sum c_m u_m$ and the problem is to find values of t such that

$$\| e^{-iHt/\hbar}\Psi - \Psi \|^2 = 4\sum |c_m|^2 \sin^2(\pi\nu_m t) < \epsilon. \tag{1}$$

As a simple example, let us assume that there is only a finite number, N, of nonvanishing $|c_m|^2$, which are all equal to $1/N$. (This simplifying assumption is not crucial, as shown below.) All the $\nu_m t$ must be very close to integers k_m, so that $\sin^2(\pi\nu_m t) \simeq \pi^2(\nu_m t - k_m)^2$ and Eq. (1) is equivalent to

$$\sum (\nu_m t - k_m)^2 < N\epsilon/4\pi^2. \tag{2}$$

Consider now the N-dimensional space with coordinates $x_1, \ldots, x_N$ where the integers $x_m = k_m$ form a unit lattice.[3] The equation

$$\sum (\nu_m t - x_m)^2 = N\epsilon/4\pi^2 \tag{3}$$

represents a sphere of radius $R = (N\epsilon)^{1/2}/2\pi$ centered at $x_m = \nu_m t$. As time passes, this sphere sweeps a cylindrical volume of radius R and length

$$(\sum \nu_m{}^2)^{1/2} t \equiv N^{1/2}\nu t, \tag{4}$$

ν being the average (rms) frequency. The cross section of this cylinder is

$$\sigma = \pi^{(N-1)/2} R^{N-1} / \Gamma[(N+1)/2]. \tag{5}$$

I expect to find, on the average, one lattice point per unit volume, i.e., typically within a time

$$t = \frac{1}{N^{1/2}\nu\sigma} \simeq \frac{1}{\nu}\left(\frac{\epsilon}{2e}\right)^{1/2}\left(\frac{2\pi}{e\epsilon}\right)^{N/2}. \tag{6}$$

For example, taking $\nu = 10^{15}\ \mathrm{sec}^{-1}$, $\epsilon = 0.1$, and $N = 50$, we obtain a recurrence time $t \simeq 1.7\times 10^{18}$ sec, more than the age of our universe.

We thus have to explain why the results are in apparent disagreement with those of Ref. 1. This cannot be due to the simplifying assumption that H is time independent and that a finite number N of states have equal weights $1/N$. Indeed, if the weights are not equal, the moving hypersphere becomes an ellipsoid and the order of magnitude of the final result remains the same. The only change in the calculation if H is time periodic is that the constants c_m of this paper are replaced by the constants r_k of Ref. 1.

The point is that one must clearly distinguish between the recurrence time of Ψ (called T in Ref. 1) and the recurrence time τ of E (the expectation value of the energy) which is illustrated in Fig. 1 of Ref. 1. For a time-independent Hamiltonian, $\tau \equiv 0$ trivially. For a time-dependent one, τ is a function of E and for some E we may have $\tau \ll E$. For example, Fig. 1 of Ref. 1 shows that the *expectation value* of the energy recurs many times. This does not prove that the *state* of the system has recurred, because there are infinitely many different states having the same expectation value for the energy. The only exception is the ground-state energy $E = 0$, corresponding to the nondegenerate ground state. This is the initial state for Fig. 1, and a value $E \simeq 0$ indeed never recurs, in agreement with the present conclusions.

In summary, it appears that quantum recurrences are not a matter of concern, unless the number of incommensurable energy levels is very small.

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