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Fluctuations in the New Inflationary Universe

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The spectrum of density perturbations is calculated in the new-inflationary-universe scenario. The main source is the quantum fluctuations of the Higgs field, which lead to fluctuations in the time at which the false vacuum energy is released. The value of $\delta\rho/\rho$ on any given length scale l, at the time when the Hubble radius $\gg l$, is estimated. This quantity is nearly scale invariant (as desired), but is unfortunately about 10^5 times too large.

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The inflationary-universe scenario was proposed by one of us[1] as a possible solution to the horizon, flatness, and monopole problems. In this scenario the universe supercools by many orders of magnitude below the critical temperature of a grand unified theory (GUT) phase transition, and in the process it exponentially expands by an enormous factor. The original version required that eventually the bubbles of the new phase would coalesce to fill the space uniformly. It was pointed out in the original paper, however, that under plausible assumptions this requirement is not fulfilled. Further studies[2,3] have shown that there is no apparent way to achieve a smooth coalescence of bubbles in the aftermath of inflation.

The hopes for the inflationary universe brightened considerably when Linde[4] and Albrecht and Steinhardt[5] proposed an alternative ending which avoids the problems described above. In this "new inflationary universe," the entire observed universe emerges from a single bubble or fluctuation. While a generic potential would lead to bubbles with far too little entropy to comprise the observed universe,[3] these authors showed that with a Coleman-Weinberg potential[6] it is very plausible that a single bubble or fluctuation can undergo enough inflation to avoid this problem. The universe expands exponentially as the Higgs field φ slowly "rolls" down the potential, and the energy is then rapidly thermalized when φ begins to oscillate about its minimum.

In this paper we will examine the consequences of the quantum fluctuations of the scalar field φ which occur during the era of exponential expansion. We will follow the evolution of these fluctuations through the time at which galactic scales come within the Hubble radius (at about 10^8 sec), and we will estimate the energy density fluctuations $\delta\rho/\rho$ at that time. According to Harrison and Zeldovich[7] this number should be about 10^{-4}, and roughly independent of scale. We find that the new inflationary universe leads to a $\delta\rho/\rho$ which is roughly independent of scale, but with a magnitude of ≈ 50. Thus, it appears that a further modification of this scenario is necessary in order to make it workable.

For concreteness we will deal with an SU(5) GUT,[8] with an adjoint Higgs field $\Phi = (\frac{2}{15})^{1/2}\varphi\,\mathrm{diag}[1, 1, 1, -\frac{3}{2}, -\frac{3}{2}]$. The Coleman-Weinberg potential then takes the form[9]

$$V(\varphi)=\tfrac{25}{16}\alpha^2[\varphi^4\ln(\varphi^2/\sigma^2)+\tfrac{1}{2}(\sigma^4-\varphi^4)]. \quad (1)$$

(We are using the flat-space potential, but we expect that gravitational corrections[10] would not significantly change our results.) We assume that the region which will evolve into the observed universe cools into a false vacuum ($\varphi\approx 0$), and it is then soon described accurately as a de Sitter space, with a Robertson-Walker $k=0$ metric

$$ds^2\equiv g_{\mu\nu}dx^\mu dx^\nu=-dt^2+R^2(t)d\vec{x}^2, \quad (2)$$

where $R(t)=e^{\chi t}$ and $\chi=[\kappa V(0)/3]^{1/2}\approx 10^{10}$ GeV (where $\kappa\equiv 8\pi G$). The scalar field φ then obeys the equation

$$\ddot{\varphi}+3\chi\dot{\varphi}=\partial V/\partial\varphi+e^{-2\chi t}\partial_i{}^2\varphi, \quad (3)$$

where the dot denotes differentiation with respect to t, $\partial_i\equiv\partial/\partial x^i$, and i is summed from 1 to 3. (We are ignoring fluctuations in the other components of Φ.) We now write $\varphi(\vec{x},t)=\varphi_0(t)+\delta\varphi(\vec{x},t)$, where $\varphi_0(t)$ is a homogeneous solution to Eq. (3).

It will be useful to have an approximate expression for $\varphi_0(t)$. For φ of order χ, we can approximate $V(\varphi)$ by

$$V(\varphi)\approx V(0)-\tfrac{1}{4}\lambda\varphi^4, \quad (4)$$

where we find [using the running coupling constant[11] of unbroken SU(5)] that $\lambda\approx\frac{1}{2}$. By neglecting the $\ddot{\varphi}_0$ term in the differential equation, one finds

$$\varphi_0{}^2(t)\approx-3\chi/2\lambda t, \quad (5)$$

where we have chosen the time at which $\varphi_0\to\infty$ to be $t=0$. φ_0 then varies approximately from 0.17χ to 1.7χ as χt varies from -100 to -1, and then φ_0 grows rapidly. Equation (3) is accurately satisfied when $-\chi t\gg 1$.

The quantity $\delta\varphi$ obeys the equation

$$\delta\ddot{\varphi}+3\chi\delta\dot{\varphi}=-\frac{\partial^2V}{\partial\varphi^2}(\varphi_0)\delta\varphi+e^{-2\chi t}\partial_i{}^2\delta\varphi. \quad (6)$$

The last term on the right-hand side decays as $e^{-2\chi t}$, and will soon become negligible. The quantity $\delta\varphi$ then obeys the same equation as $\dot{\varphi}_0$, and the presence of the damping term implies that any two solutions approach a time-independent ratio at large times.[12] Thus, at large times one can write

$$\delta\varphi(\vec{x},t)\to-\delta\tau(\vec{x})\dot{\varphi}_0(t), \quad (7)$$

and then to first order in $\delta\tau$,

$$\varphi(\vec{x},t)\to\varphi_0(t-\delta\tau(\vec{x})). \quad (8)$$

Thus, the effect of the fluctuations is simply to produce a position-dependent time delay in the evolution of $\varphi_0(t)$.

We will return later to estimate the magnitude of the fluctuation $\delta\tau(\vec{x})$. We will first assume that $\delta\tau$ is given, and we will calculate $\delta\rho/\rho$. To simplify the analysis, we will make two approximations: (i) The space-time metric will be taken as exactly de Sitter until $t=\delta\tau(\vec{x})$; the fluctuations in $T_{\mu\nu}$ will be considered to be negligible compared with the false vacuum energy density. (ii) The transition from false vacuum (with pressure $p=-\rho$) to radiation ($p=\frac{1}{3}\rho$) will be assumed to take place instantaneously at $t=\delta\tau(\vec{x})$. In a subsequent paper we will show by means of a more complicated analysis that the error resulting from these approximations is totally negligible.

To understand the transition from false vacuum to radiation, we will define a new coordinate system by $t'\equiv t-\delta\tau(\vec{x})$, $\vec{x}'\equiv\vec{x}$. In this coordinate system the transition occurs at $t'=0$, but the perturbations are recorded in the metric

$$g_{\mu\nu}'=\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\lambda\sigma}. \quad (9)$$

The energy-momentum tensor for $t'>0$ is taken to be that of an ideal radiation fluid:

$$T^{\mu\nu}=pg^{\mu\nu}+(p+\rho)u^\mu u^\nu, \quad (10)$$

where $p=\frac{1}{3}\rho$ and u^μ denotes the local fluid velocity, with $u^2\equiv-1$. The values of ρ and u^μ at $t'=+\epsilon$ are completely determined by $D_\nu T^{\mu\nu}=0$, where D_ν denotes the covariant derivative. Note that u^μ is undefined for $t'<0$, but we can define it to be continuous at $t'=0$. Then

$$\begin{aligned}D_\nu T_0{}^\nu&=[\Delta p+u_0u^0(\Delta p+\Delta\rho)]\delta(t')+\ldots,\\ D_\nu T_i{}^\nu&=u_iu^0(\Delta p+\Delta\rho)\delta(t')+\ldots,\end{aligned} \quad (11)$$

where $\Delta p=p(t'=+\epsilon)-p(t'=-\epsilon)$, $\Delta\rho$ is defined similarly, and only the $\delta(t')$ terms have been written out. In the spirit of our first-order perturbation analysis, we assume that the $t'=0$ hypersurface is spacelike; it follows that $u^0\neq 0$. The vanishing of Eqs. (11) then implies $\Delta\rho=0$, $u_i=0$; i.e., u^μ must be orthogonal to the $t'=0$ hypersurface. We take $u^0>0$.

To follow the evolution of the density-wave perturbations for $t'>0$, we will use the formalism of Olson.[13] To use the formalism, we must compute the initial (i.e., $t'=0$) values of $\theta\equiv h^{\alpha\beta}D_\alpha u_\beta$ and $\operatorname{div}X\equiv D_\alpha(h^{\alpha\beta}D_\beta\rho)$, where $h^{\alpha\beta}\equiv g^{\alpha\beta}+u^\alpha u^\beta$ is the operator which projects onto the space orthogonal to u^μ. A straightforward calculation (to first order in $\delta\tau$) yields $\theta = \Gamma_{i i}{}^{0} = 3\chi + \partial_i{}^2\delta\tau$, and $\mathrm{div}X = 0$. The quantity S is then defined by $3\kappa\rho \equiv \theta^2(1+S)$; so $S(t'=0) = -\frac{2}{3}\chi^{-1}\partial_i{}^2\delta\tau$.

For $t'>0$ we introduce another time variable $t'' \equiv t' + \frac{1}{2}\chi^{-1}$; the unperturbed metric then has the form of Eq. (2), with $R(t'') = (2\chi t'')^{1/2}$ and $\rho = 3/4\kappa t''^2$. (Note that at $t'' = \frac{1}{2}\chi^{-1}$, $R=1$ and $\dot{R} = \chi$, as required by continuity.) The evolution of the perturbations is then governed by the equations[13,14]

$$\dot{S} = (t'')^{-1}S + \tfrac{4}{9}\kappa t''^3\,\mathrm{div}X, \qquad (\mathrm{div}X)^{\boldsymbol{\cdot}} = -\frac{7}{2t''}\,\mathrm{div}X + \frac{3}{8\kappa\chi t''^4}\,\partial_i{}^2 S. \tag{12}$$

If the functions $\delta\tau(\vec{x})$ and $S(\vec{x})$ are Fourier expanded as $f(\vec{x}) = \int d^3k\, \exp(i\vec{k}\cdot\vec{x})\tilde{f}(\vec{k})$, then Eqs. (12) can be solved by introducing a dimensionless time variable $x \equiv (2k^2t''/3\chi)^{1/2}$ and eliminating $\mathrm{div}X$ to obtain

$$x^2\frac{d^2\tilde{S}}{dx^2} - 2x\frac{d\tilde{S}}{dx} + (2+x^2)\tilde{S} = 0, \tag{13}$$

which has solutions $x\sin x$ and $x\cos x$.

The initial conditions are fixed at $x_0 = k\chi^{-1}/\sqrt{3}$, and at this time $\tilde{S} = \frac{2}{3}k^2\chi^{-1}\delta\tilde{\tau}$ and $x\,d\tilde{S}/dx = 2\tilde{S}$. To relate x_0 to current length scales, note that $R(t)$ was normalized to unity immediately after the phase transition, when the temperature $T \sim 10^{14}$ GeV. Taking $RT \sim$ const, one finds $R \sim 10^{27}$ today. Identifying k^{-1} with a coordinate distance l, one has $k\chi^{-1} = R\chi^{-1}/l_{\mathrm{phys}}$, where l_{phys} is the current length and $R\chi^{-1} \sim 10$ m $\sim 10^{-15}$ light-yr. Thus, for a galactic scale of, say, 10^6 light-yr, $x_0 \sim 10^{-21}$. The desired solution is then

$$\tilde{S} = 2\chi\delta\tilde{\tau}x\sin x[1 + O(x_0{}^2)]. \tag{14}$$

Finally, we determine $\delta\tilde{\rho}$ by using the formula[13] $(t''\delta\rho/\rho)^{\boldsymbol{\cdot}} = S$, which can be integrated to give

$$\frac{\delta\tilde{\rho}}{\rho} = 4\chi\delta\tilde{\tau}\left(-\cos x + 2\frac{x\sin x + \cos x - 1}{x^2}\right). \tag{15}$$

Thus, $\delta\tilde{\rho}/\rho$ grows as x^2 for $x \ll 1$, but for large x (when the wavelength of the perturbation becomes small compared with the Hubble radius) it begins to oscillate with a root-mean-square value

$$\delta\tilde{\rho}_{\mathrm{rms}}/\rho = 2\sqrt{2}\chi\delta\tilde{\tau}. \tag{16}$$

We now return to the question of estimating the magnitude of the fluctuations in $\delta\tau(\vec{x})$. Given any stochastic function $f(\vec{x})$ with correlation functions which are invariant under translations and rotations, one can measure the mean fluctuations of wave number $\vec{k}$ by[15]

$$\Delta f(\vec{k}) \equiv \left((2\pi)^{-3}k^3\int d^3x\,\exp(i\vec{k}\cdot\vec{x})\langle f(\vec{x})f(\vec{0})\rangle\right)^{1/2}. \tag{17}$$

To estimate $\Delta\tau(\vec{k})$ consider again Eq. (6). Let $t^*(k)$ denote the time at which the two terms on the right-hand side are equal in magnitude. Using Eqs. (4) and (5), one finds

$$-\chi t^* \approx \ln(\chi k^{-1}) - \tfrac{1}{2}\ln(-\tfrac{2}{9}\chi t^*) \approx \ln(\chi k^{-1}), \tag{18}$$

where we are interested in values of χk^{-1} of order 10^{21}. For $t \ll t^*$ the potential term is negligible, and the two-point correlation function for $\delta\varphi$ is determined by the quantum theory of a free massless scalar field in de Sitter space[16]:

$$\Delta\varphi(\vec{k},t) = \{(\chi^2/16\pi^3)[1 + (\chi^{-1}k)^2e^{-2\chi t}]\}^{1/2}. \tag{19}$$

On the other hand, the time lag $\delta\tau(\vec{x})$ is determined by Eq. (7) for $t \gg t^*$. Thus, we will estimate $\Delta\tau$ by assuming that Eqs. (19) and (7) are both reasonable approximations at t^*. Then

$$\Delta\varphi(\vec{k},t^*) \approx \chi/(16\pi^3)^{1/2} \tag{20}$$

and

$$\dot{\varphi}_0(t^*) \approx (3/8\lambda)^{1/2}\chi^2/\ln^{3/2}(\chi k^{-1}). \tag{21}$$

In Eq. (20) we made use of the fact that

$$(\chi^{-1}k)^2\exp(-2\chi t^*) \approx 9/[2\ln(\chi k^{-1})] \ll 1$$

for the cases of interest. It then follows that

$$\Delta\rho/\rho = 2\sqrt{2}\,\chi\Delta\tau = (4\lambda/3\pi^3)^{1/2}\ln^{3/2}(\chi k^{-1}). \tag{22}$$

For a galactic scale of 10^6 light-yr, Eq. (22) gives $\Delta\rho/\rho \approx 49$. On a scale of 10^{10} light-yr we find $\Delta\rho/\rho \approx 64$. Thus the spectrum of fluctuations is sufficiently flat—it is just the magnitude which is wrong.

While this discrepancy is of course disappointing, one should remember that this is the first model with sufficient detail to allow an estimate from first principles of the magnitude of $\delta\rho/\rho$. (Calculations based on cosmic strings[17] require the *assumption* of an initial space which is perfectly homogeneous.) The estimate is somewhat model dependent, and it is conceivable that the desired number can be obtained by modifying the scenario and/or the underlying particle theory.

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Note added.—After this calculation was completed we learned that Starobinsky[18] had already obtained similar conclusions. Hawking[19] and Bardeen, Steinhardt, and Turner[20] have announced results which are also in agreement. We have also received a preprint by Pagels[21] which obtains a much smaller answer; however, we believe that the effects which he considers are subdominant.

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