Exact Critical Point and Critical Exponents of O(n) Models in Two Dimensions

Bernard Nienhuis
James Franck Institute, University of Chicago, Chicago Illinois 60637, and Department of Electronics, Weizmann Institute of Science, Rehovot, Israel
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A two-dimensional n-component spin model with cubic or isotropic symmetry is mapped onto a solid-on-solid model. Subject to some plausible assumptions this leads to an analytic calculation of the critical point and some critical indices for $-2 \leq n \leq 2$.

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Two-dimensional classical O(n) spin models,[1] alias n-vector models, have been discussed in a variety of contexts. The O(2) model describes the superfluid-to-normal transition of helium films.[2] The n-vector model with cubic symmetry breaking provides the universality classes of many ordering transitions of monolayers. Examples on a square substrate are the ordering into a 2×2 or 2×1 array ($n=2$) and into a $\sqrt{2}\times\sqrt{2}$ array ($n=1$).[3] In the limit $n\to 0$ the O(n) model describes the scaling behavior of long polymer chains.[4]

The O(n) phase transition in general dimension d has been studied theoretically by means of expansion about special values of n or d. The model served as a paradigm for the $4-\epsilon$ expansion.[5] The critical exponents for $n>2$ have also been expanded[6] in $d-2$. For general d the values $n=-2$ (Gaussian model) and $n=\infty$ (spherical model) have permitted exact calculations,[7] the latter serving as a basis for expansions in $1/n$. Thus the section of the n-d plane, $2\leq d\leq 4$ and $n\geq -2$, which contains most of the physically applicable models is almost encompassed by exact results.[8] For $d=2$ and $-2\leq n\leq 2$ this Letter presents some results which complete this enclosure. The main conclusions are summarized below.

To define notation let the singular part of the free energy of a two-dimensional O(n) model be $\epsilon^{2\nu}(f_0+f_1\epsilon^{\theta}+\ldots)$, where f_0 and f_1 are analytic functions of $\epsilon=|T-T_c|/T_c$, and the ellipsis represents singular contributions of higher order in ϵ. The exponents ν and θ are computed for

$$n=-2\cos(2\pi/t), \tag{1}$$

and are

$$1/\nu\equiv y_{T1}=4-2t, \tag{2}$$

$$-\theta/\nu\equiv y_{T2}=6-6t, \tag{3}$$

with $1\leq t\leq 2$. At the critical point the spin-spin correlation function decays with distance r as $r^{-\eta}$. The exponent I conjecture is

$$2-\eta/2\equiv y_H=1+3/4t+t/4. \tag{4}$$

In the low-temperature phase (for $n>1$) there is still algebraic decay of the spin-spin correlation function[6] with the exponent also defined by Eqs. (1) and (4), but now on the branch $t>2$. These results are obtained from an analysis of an O(n) model on a honeycomb lattice, for which the critical point is calculated as well.

Consider an n-component spin model with partition integral

$$Z_{\mathrm{O}(n)}=\int\prod_{\langle ij\rangle}(1+x\vec{S}_i\cdot\vec{S}_j)\prod_k W(\vec{S}_k)d^2S_k. \tag{5}$$

The first product is over nearest-neighbor pairs of sites of the honeycomb lattice with free boundaries. The weight function $W(\vec{S})$ is either isotropic, i.e., invariant under arbitrary rotations of $\vec{S}$, or cubic, i.e., invariant under permutations and inversions of the components of $\vec{S}$. The length of the spins and the weight function are normalized such that $\int W(\vec{S})d^2S=1$ and $\int W(\vec{S})\vec{S}\cdot\vec{S}d^2S=n$. It has been shown recently[9] for both cubic and isotropic W that $Z_{\mathrm{O}(n)}$ can be expanded in diagrams consisting of loops on the honeycomb lattice:

$$Z_{\mathrm{O}(n)}=\sum_G x^L n^c, \tag{6}$$

where c is the number of loops in diagram G and L their combined length.

Now consider a triangular lattice with each site having an integer height variable M, so that no two adjacent heights differ by more than one. Any basic triangle can have all three of its M's equal or one different and two equal. When all three heights are equal the triangle has weight one. If one of the three M's, say M_1, is different from the other two, M_2 and M_3, then the triangle has weight $xe^{i\alpha}$ if $M_1>M_2=M_3$ or $xe^{-i\alpha}$ if $M_1<M_2=M_3$. Examples of these configurations are shown in

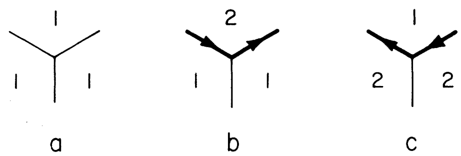

FIG. 1. Configurations of integers on a triangle. The weights a, b, and c are invariant if the configurations are rotated in space, or if all heights are changed by the same amount.

Fig. 1 with weights

$$a,\ b,\ c = 1,\ xe^{i\alpha},\ xe^{-i\alpha}. \tag{7}$$

This defines a solid-on-solid (SOS) model. The partition sum can be expanded in diagrams representing domain walls between regions of different height. An arrow is placed on the domain wall with the higher M to the left of it, as indicated in Fig. 1. The factors $e^{i\alpha}$ and $e^{-i\alpha}$ are then associated with left and right turns of the oriented domain wall. The partition sum is

$$Z_{\mathrm{SOS}} = \sum\nolimits_{OG} x^L e^{i\alpha(l-r)}, \tag{8}$$

where the summation is over graphs consisting of oriented loops on the honeycomb lattice with combined length L and a total of l left and r right turns.

The summation over the orientation can be performed first and for each loop independently. Since each loop makes exactly one full turn in the plane it contributes either 6 or -6 to $l-r$. Hence

$$Z_{\mathrm{SOS}} = \sum\nolimits_{G} x^L (e^{6i\alpha} + e^{-6i\alpha})^c, \tag{9}$$

where c is the number of loops or separate domain walls. Clearly $Z_{\mathrm{O}(n)}$ equals Z_{SOS} if

$$n = 2\cos(6\alpha). \tag{10}$$

A third model to be considered is a six-vertex (6V) model on a Kagomé lattice[10] shown in Fig. 2. Arrows are placed on all the edges of the lattice, and weights ω_1 through ω_6 are associated with six allowed arrow configurations, shown in Fig. 3. The weights are

$$\omega_1, \ldots, \omega_6 = \tau,\ \tau,\ 1,\ 1,\ u+iv,\ u-iv. \tag{11}$$

The edges of the lattice can be interpreted as domain walls in a SOS model, like in Fig. 1, with the greater height to the left of the arrow. Let integers M be placed on the hexagonal faces of the Kagomé lattice and half-integer variables m on the triangular faces, so that each m differs by $\frac{1}{2}$ from the adjacent M. Examples of configurations of M and m are shown in Fig. 3.

A special case of (11),

$$\omega_1, \ldots, \omega_6 = \tau,\ \tau,\ 1,\ 1,\ e^{-2i\alpha},\ e^{2i\alpha}, \tag{12}$$

represents the following interaction of M and m. A factor τ is associated with each unequal pair of adjacent M. Each nearest-neighbor pair of M and m contributes $e^{i\alpha}$ if $m > M$ or $e^{-i\alpha}$ if $m < M$. Therefore the m do not interact among themselves and can be summed over. This leaves an effective interaction among the M summarized in the configurations of Fig. 1, with weights a, b, $c = 2\cos(3\alpha)$, $\tau e^{i\alpha}$, $\tau e^{-i\alpha}$. Taking the first weight as a normalization this is equivalent to (7) for

$$\tau = 2x\cos(3\alpha). \tag{13}$$

Thus the 6V model (12) is equivalent to the SOS model (7) and hence to the O(n) model (4).

Another special case of (11),

$$\omega_1, \ldots, \omega_6 = \tau,\ \tau,\ 1,\ 1,\ e^{2i\varphi} + \tau e^{-i\varphi},\ e^{-2i\varphi} + \tau e^{i\varphi}, \tag{14}$$

is equivalent[10] to a Potts model on a triangular lattice. The Potts Hamiltonian is

$$-\beta H = \sum_{\langle ij\rangle} J\delta_{\sigma_i\sigma_j}, \tag{15}$$

summed over the edges of the triangular lattice,

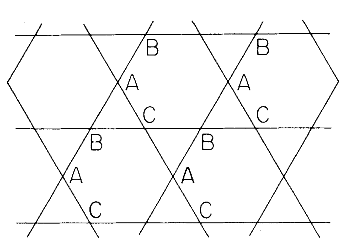

FIG. 2. The Kagomé lattice. Vertices of the three sublattices A, B, and C differ by rotation over $\pm 2\pi/3$.

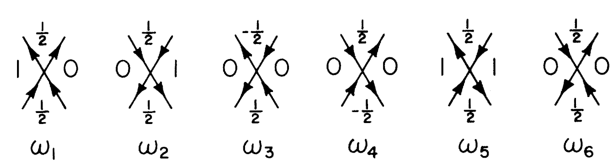

FIG. 3. The configuration of the six-vertex model as they appear on sublattice A (see Fig. 2). Rotation of the vertex over $\pm 2\pi/3$ to get configuration on sublattice B or C leaves the weight invariant. The numbers placed around the vertices are M and m as explained in the text.

with variables $\sigma_i = 1, 2, \ldots, q$ and $q = 2 + 2\cos(6\varphi)$. J is defined by $e^J - 1 = 2\cos(3\varphi)/\tau$.

The weights (14) and (12) evidently coincide if

$$\tau = -2\cos(3\varphi), \quad \alpha = 2\varphi + \pi/2. \tag{16}$$

Thus the three-dimensional parameter space of the 6V model (11) contains two-parameter subspaces equivalent to the O(n) model (12) and to the Potts model (14), which intersect in Eq. (16).

It has been recognized that the critical behavior of SOS[11] and 6V models[12] can be understood as that of a Gaussian model with spin-wave fields, which in turn is equivalent[11] to a Coulomb gas (CG). The 6V model (11) behaves like[13] CG with fugacity z_+ for positive and z_- for negative unit charges and temperature $1/t$ low enough so that multiple charges are bound. The renormalization-group equations read[11]

$$dt/dl = -t^2 z_+ z_-, \tag{17}$$

$$dz_+/dl = (2-t)z_+ + z_+^2 z_- g(t), \tag{18}$$

$$dz_-/dl = (2-t)z_-. \tag{19}$$

The function g need not be specified. These equations hold to first order in z_-, to zeroth order in the fugacities of multiple charges, and as such to all orders in z_+. When $t<2$ the fugacities z_+ and z_- grow in magnitude under renormalization. If $z_+ z_- > 0$ they continue to grow indefinitely. However, if $z_+ z_- < 0$ then t increases which eventually makes z_+ and z_- irrelevant and causes them to vanish. Therefore, the locus $z_- = 0$ is critical, separating regimes that renormalize towards the Gaussian line ($z_+ = z_- = 0$) and away from it. It has been shown[13] that the Potts subspace is tangential to this separatrix at the Gaussian line. If one assumes that the Potts model renormalizes onto itself, the subspace (14) must coincide[14,15] with the locus $z_- = 0$. This implies that the intersection (16) represents a singular line of the O(n) subspace. Hence the critical point of the O(n) model (4) follows from substituting Eq. (16) into Eqs. (10) and (13),

$$x_c = [2 + (2-n)^{1/2}]^{-1/2}. \tag{20}$$

This reduces to the known result for[16] $n = 1$ and[9] 2, and agrees with numerical estimates[17] for $n = 0$. The regime $x > x_c$ renormalizes towards the Gaussian line, which governs the critical fluctuations of the Goldstone modes.[6]

The thermal critical exponent of the O(n) model is the one associated with z_-. As a consequence of charge neutrality, however, the free energy depends on z_- through $z_+ z_-$. Therefore, as z_- vanishes at finite z_+ the effective exponent is that of the product $z_+ z_-$, namely, $y_{T1} = 4 - 2t$ [Eq. (2)]. A similar argument[15] yields for a correction to scaling exponent $y_{T2} = 6 - 6t$ [Eq. (3)]. The eight-vertex exponent y_{8V} is known exactly ($y_{8V} = 6\varphi/\pi$),[18] and can also be expressed in the CG language ($y_{8V} = 2 - 1/t$), in which it is associated with magnetic charges.[11] This gives a relation between φ and t, which substituted in Eqs. (16) and (10) yields t as a multivalued function of n, Eq. (1). The branch $1 \le t \le 2$ gives the critical-point behavior, and the branch $t > 2$, where z_+ and z_- are irrelevant, pertains to the low-temperature phase of the O(n) model.

Equation (2) for exponent ν was recently conjectured,[19] on the basis of its known values for $n = -2$, 1, and 2, together with the assumption of a simple dependence on n. The success of such arguments suggests a similar guess for the magnetic eigenvalue, Eq. (4). This exponent predicts how the longitudinal susceptibility in the low-temperature phase of the O($n > 1$) model diverges as the field vanishes. It is remarkable that Eq. (4) is almost identical to the conjectured magnetic eigenvalue of the Potts model[20] $y_{HP} = 1 + 3t/4 + 1/4t$.

The results of this Letter are derived for a particular, somewhat unphysical model, which is hopefully a faithful representative of the O(n) universality class. A positive indication of this is that the same exponents are derived for another O(n) model on a square lattice.[15] This model turns out to be identical to a dilute n^2-state Potts model at its transition, with y_{T1} corresponding to the second tricritical exponent[13] of this Potts model.

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Geometrization of Quantum Mechanics and the New Interpretation of the Scalar Product in Hilbert Space

E. Prugovečki
Department of Mathematics, University of Toronto, Toronto, M5S 1A1 Canada

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It is shown that the new interpretation of the scalar product in Hilbert space recently proposed by Aharanov, Albert, and Au is in fact the one underlying the stochastic phase-space formulation of nonrelativistic quantum mechanics. The extrapolation of this interpretation to the relativistic domain and its relationship to the program of geometrizing quantum theory and quantizing space-time are discussed.

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In a recently published Letter[1] Aharanov, Albert, and Au have presented a "new" interpretation of the scalar product in Hilbert space. This Letter was soon followed by a Letter of O'Connell and Rajagopal[2] pointing out the equivalence of this interpretation (via recent results by O'Connell and Wigner[3,4]) to Wigner's pioneering work[5] on phase-space distributions in nonrelativistic quantum mechanics. In the present note we intend to point out first, that in fact this "new" interpretation is the one underlying the stochastic phase-space approach to nonrelativistic quantum mechanics,[6,7] and, second, but more important, that this entire approach extends[8] to the relativistic domain, where it leads[9] to a consistent operationally based concept of quantum space-time, or, equivalently, to a new approach to the geometrizing[10-12] of quantum mechanics.

For the sake of simplicity, let us consider only single quantum particles of spin 0. The nonrelativistic quantum mechanics of such particles can be formulated over spaces of exact values, such as the conventional configuration or momentum spaces R^3, as well as[14] over spaces Γ_φ of stochastic phase-space values $(\vec{q},\chi_{\vec{q}})\times(\vec{p},\hat{\chi}_{\vec{p}})$—these last spaces being special cases of Menger-Wald statistical metric spaces.[15] In the former case the measurements of position *or* momentum are assumed to be perfectly accurate, whereas in the second case the measured values are stochastically spread out (so that the uncertainty principle is not violated[7]), and in the optimal cases they represent measurements with extended test particles of normalized proper wave function φ (in the sense of Landé[16] and Born[17]).

Mathematically, these nonrelativistic stochastic phase-space representations can be arrived at by considering[8,9] unitary ray representations of the Galilei group on the Hilbert space $L^2(\Gamma)$ of functions $\psi(\vec{q},\vec{p})$ with inner product

$$\langle\psi_1|\psi_2\rangle=\int_\Gamma \psi_1^*(\vec{q},\vec{p})\psi_2(\vec{q},\vec{p})d^3q\,d^3p. \quad (1)$$

