PHYSICAL REVIEW LETTERS

VOLUME 49 | 11 OCTOBER 1982 | NUMBER 15

Improved Perfect-Fluid Energy-Momentum Tensor with Spin in Einstein-Cartan Space-Time

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(Received 17 August 1982)

An improved perfect-fluid energy-momentum tensor including spin and torsion is presented with use of a Lagrangian variational principle based upon the tetrad formalism of Halbwach and the method of constraints of Ray.

PACS numbers: 04.20.Fy, 04.20.Me, 04.50.+h

We have improved the description of a perfect fluid by including spin and torsion through use of a Lagrangian variational principle based upon the formalism of Halbwach[1] and the method of constraints of Ray.[2] Ambiguities in the constraints[3] in the presence of torsion are resolved in favor of the perfect-fluid description with mass conservation. For consistency, we have also extended the thermodynamic description of the internal energy of a fluid so as to include spin as a thermodynamic variable.

In this work, the description of the spin is classical in that it is intrinsic but not quantized. Our approach in this matter is similar, for example, to the work of Bailey and Israel[4] where the fluid particles, which have intrinsic spin, may be galaxies or clusters of galaxies. We do not resort to the elementary particles of these objects and the "ferromagnetic alignment" of their quantum spins in order to describe a fluid with spin. Physically this means that the equation of motion for the spin tensor is a modified Fermi-Walker transport equation[5] and arises as a direct result of including spin as an intrinsic variable in the thermodynamic description of the internal energy. Also the variables in our description are classical variables throughout and not microscopic fields.

The tetrad, $a^{\mu}{}_{j}$ ($\mu = 1, \ldots, 4$ are anholonomic coordinates, $j = 0, 1, 2, 3$ are holonomic), is chosen such that $a^{4j} = u^j$ is the four velocity, the spin of a fluid particle is

$$s_{ij} = \kappa(x)(a^1{}_i a^2{}_j - a^1{}_j a^2{}_i), \qquad (1)$$

where $\kappa(x)$ is a scalar function, and the angular velocity is

$$\omega_{ij} = \tfrac{1}{2}(\dot{a}^{\mu}{}_i a_{\mu j} - \dot{a}^{\mu}{}_j a_{\mu i}), \qquad (2)$$

where $\dot{a}_{\mu i} = a_{\mu i;k} u^k$.

Our Lagrangian density takes the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_H, \qquad (3)$$

where

$$\mathcal{L}_G = ec_1R - e\rho[1+\epsilon(\rho, s, s_{ij})] + e\lambda_1(g_{ij}u^iu^j+1) + \lambda_2(eu^i)_{,i} + e\lambda_3 X_{,i}u^i + e\lambda_4 s_{,i}u^i \tag{4}$$

and

$$\mathcal{L}_H = e[-\rho\kappa(x)a^{1j}\dot{a}^2{}_i + \lambda^{11}(a^1{}_i a^{2i} - 1) + \lambda^{22}(a^{2i}a^2{}_i - 1) + 2\lambda^{12}a^{1i}a^2{}_i + 2\lambda^{14}a^{1i}u_i + 2\lambda^{24}a^{2i}u_i], \tag{5}$$

where e is the square root of the determinant of the metric, $c_1 = 1/16\pi G$, G is the gravitational constant, R is the scalar curvature, ρ is the mass density of the fluid, X is a fluid particle number, s is the specific entropy, and the various λ's are the constraints which maintain the conservation laws for the fluid and ensure orthonormality of the tetrads. For convenience, we take $c=1$. To supplement the Lagrangian (3), the thermodynamic laws are contained in

$$d\epsilon = T\,ds - P\,d(1/\rho) + \tfrac{1}{2}\omega_{ij}ds^{ij}, \tag{6}$$

where P is the pressure and T is the temperature.

The portion of the Lagrangian given by $\mathcal{L}_G$ leads to the usual description of a perfect fluid in general relativity when the spin variable is suppressed. For details see Ref. 2. The extension to a connection with torsion, i.e., an Einstein-Cartan or U_4 space-time, is easily obtained.[6] $\mathcal{L}_H$ leads to Halbwach's treatment of spin in special relativity. When the spin variable in ϵ is suppressed and torsion is held equal to zero, the combination of $\mathcal{L}_G + \mathcal{L}_H$ leads to a general relativistic treatment of spin and a perfect-fluid energy-momentum tensor corrected for spin. Details will be reported elsewhere.[7] Including both the spin variables in ϵ along with an asymmetric connection, i.e., torsion, the variation of $\mathcal{L}$ with respect to g_{ij}, $S_{ij}{}^k$, ρ, X, s, u^i, s_{ij}, $a^1{}_i$, $a^2{}_i$, and the various λ's yields a description of spinning fluids in the Einstein-Cartan space-time.

We digress for a moment and consider the term involving the λ_2 constraint. In general relativity this can be written in two equivalent ways:

$$e(\rho u^i)_{;i} \tag{7}$$

or

$$(e\rho u^i)_{,i}. \tag{8}$$

But if we extend them to a U_4 space-time, we get inequivalent results.[3] The constraint for the first case becomes

$$e\nabla_i(\rho u^i) = 0, \tag{9}$$

where ∇_i is the covariant derivative for the full U_4 connection; and for the second

$$(e\rho u^i)_{,i} = \nabla^*{}_i(e\rho u^i) \equiv e\nabla_i(\rho u^i) + 2eS_{ij}{}^j\rho u^i = 0, \tag{10}$$

where the second line defines the "star" derivative in terms of the trace of the torsion tensor

$$S_{ij}{}^k = \Gamma_{[ij]}{}^k \tag{11}$$

given by the antisymmetric part of the affine connection $\Gamma_{ij}{}^k$. Applying Gauss's law to (9) for the region Σ between two spacelike hypersurfaces σ_1 and σ_2 enclosing all matter, we find

$$M(\sigma_2) - M(\sigma_1) = 2\int eS_{ij}{}^j\rho u^i d^4x, \tag{12}$$

where

$$M(\sigma) = \int_\sigma eu^i d^3\Sigma_i. \tag{13}$$

This leads to the novel interpretation that the torsion vector $S_{ij}{}^j$ is associated with mass creation (annihilation) of the system and may have applications for mass-creation cosmological theories. An alternative interpretation can be seen for the special case when the trace-free portion of the torsion vanishes. The torsion vector is then proportional to the Weyl vector[8] and can be interpreted as the precursor of a volume-nonpreserving space-time. In the variable mass theory, torsion is not very closely associated with spin. For example, from the constraint (9), we find, upon varying $\mathcal{L}$, the following relation between torsion and spin:

$$4c_1T^{ijk} = 2\lambda_2 u^{[i}g^{j]k} + s^{ij}u^k, \tag{14}$$

where the modified torsion tensor

$$T_{ij}{}^k = S_{ij}{}^k + 2\delta_{[i}{}^k S_{j]x}{}^x. \tag{15}$$

On the other hand for the constraint (10),

$$4c_1T^{ijk} = s^{ij}u^k. \tag{16}$$

Equation (16) is in the form of a spinning Weyssenhoff fluid. Note also that this occurs because the torsion vector vanishes in (16). For our choice of the Lagrangian $\mathcal{L}$, the modified torsion is proportional to the spin angular momentum tensor.[9] Thus, coupled with the constraint (10), that is mass conservation, $\mathcal{L}$ *leads to the first* *satisfactory variational-principle description of a Weyssenhoff fluid.* One must note the strong connection between torsion and spin. From (16), if the spin vanishes, then torsion vanishes. The inverse of (16) gives

$$S^{ij} = -(4c_1/\rho)u_k T^{ijk}, \quad (17)$$

so that the converse is also true. For the mass nonconservation constant, Eq. (14) shows that even when the spin vanishes, there is still torsion. Thus, as stated above, spin and torsion are not closely linked in that theory.

Upon variation of $\mathcal{L}$ with respect to the metric, using the thermodynamic law (6), and after eliminating the constraints, we finally obtain the improved energy-momentum tensor for a spinning perfect fluid in a U_4 space-time:

$$T_s{}^{ij} = \{\rho(1+\epsilon+P/\rho)u^i u^j + Pg^{ij}\} + \{\rho\dot{u}_k u^{(i} s^{j)k} + \nabla^*{}_k[\rho u^{(i} s^{j)k}]\} - \{\rho\omega_k{}^{(i} s^{j)k}\}. \quad (18)$$

Details of the calculation will be given elsewhere.[10] The first expression in curly braces on the right-hand side of (18) for $T_s{}^{ij}$ is the usual perfect-fluid energy-momentum tensor; the second is the spinning fluid extension of Halbwach's Lagrangian to U_4 space-time (general relativity if torsion is neglected); and the third is the correction resulting from the extension of the thermodynamic law (6) for spin. This last term has an interesting consequence for the equation of motion of the spin, which becomes

$$\frac{D}{d\lambda} s_{ij} - (\Omega_{ik} - \omega_{ik}) s^k{}_j - (\Omega_{jk} - \omega_{jk}) s_i{}^k = 0, \quad (19)$$

where $\Omega_{ik} = u_i \dot{u}_k - \dot{u}_i u_k$ is the usual Fermi-Walker transport tensor.[5] Because of (2), Eq. (19) is equivalent to

$$\frac{D}{d\lambda} s_{ij} - \tfrac{1}{2}\Omega_{ik} s^k{}_j - \tfrac{1}{2}\Omega_{jk} s_i{}^k = 0, \quad (20)$$

so that the motion of the spin is governed by one-half the magnitude of the Fermi-Walker transport tensor. This result should be manifest beyond the astrophysical setting presented here since it follows mainly from the special confluence of Halbwach's special relativistic treatment of spin and from thermodynamics.

Although it is possible to obtain the improved energy-momentum tensor, $T_s{}^{ij}$, by mating Halbwach's formalism with general relativity, we feel that this approach is not complete. We have difficulty matching the source of the spin angular momentum tensor to the spin. However, we find that the natural union occurs in a U_4 space-time. Here we find that the torsion and spin are so intimately related that you can only have one with the other. If the fluid has spin, then it has torsion and vice versa. The importance of this observation resides in the fact that the natural framework for gravitation and spin is a U_4 space-time,[9] and that when spin vanishes (or is neglected), general relativity results. *We have in effect obtained the geometrization of spin.*

With this interpretation, the improved energy-momentum tensor, $T_s{}^{ij}$, should be directly applicable to cosmological problems involving spinning matter in the early universe as well as to the unsolved problem of the interior Kerr solution. In the latter case, the outside portion should reduce to general relativity whereas the interior region must be a U_4 space-time with *both* spin and torsion. Since $T_s{}^{ij}$ has additional terms related to the spin density of the fluid, it seems likely that one can now match boundary terms between the interior and exterior solutions.[11]

One of us (J.R.R.) was the recipient of a NASA/ASEE Summer Faculty Fellowship.

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