Renormalization-Group Transformation for the Anderson Transition

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A lattice model of a d-dimensional disordered system is considered. The model is studied by a Migdal-Kadanoff renormalization-group transformation. In particular, localization in a macroscopically inhomogeneous medium is considered, and it is demonstrated that macroscopic disorder (above the percolation threshold) does not affect the critical exponents of the Anderson transition.

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Some time ago Abrahams *et al.*[1] developed a scaling theory of localization for noninteracting electrons. The scaling parameter in their theory is the dimensionless conductance $g(L)=G(L)/(e^2/\hbar)$ at length scale L. Here $G(L)$ is the zero-temperature dc conductance of a hypercube of size L in d dimensions. For $d>2$, the scaling theory of Ref. 1 predicts a metal-insulator transition. At the transition the conductance $g(L)$ of a sample is independent of sample size L and it assumes some critical value g_c (i.e., the conductivity, $\sigma_c=\lim_{L\to\infty}L^{2-d}g_c$, is zero). This value is achieved when the Fermi level E_F coincides with the mobility edge E_c. Near (above) E_c the conductivity $\sigma(E_F)$ obeys a power law $(E_F-E_c)^t$, while the correlation (localization) length ξ diverges as $(E_F-E_c)^{-\nu}$. Exponents t and ν are related by[1,2] $t=(d-2)\nu$. Near two dimensions (i.e., when $d-2\equiv\epsilon\ll 1$) $\nu=1/\epsilon$, i.e., $t=1$.[3-5] Various numerical estimates of the exponent ν in three dimensions range from[6] $\nu\approx 0.66$ to[7] $1.25<\nu<1.75$ (Ref. 7 contains a summary of available estimates).

The purpose of this Letter is to introduce a d-dimensional lattice model of a disordered medium, and to study this model by a simple renormalization-group (RG) transformation. Both microscopic (quantum) and macroscopic (classical) disorder can be incorporated into the model, so that the interplay between quantum localization and classical percolation can be studied. The model is based on a scattering formalism for the quantum transport problem—a formalism which has been successfully used in recent analytical and numerical studies of localization (see Refs. 8-13 and references therein).

I consider a d-dimensional hypercubic lattice. Each site of the lattice is occupied by a random scatterer with $2d$ incoming and $2d$ outgoing channels (see Fig. 1 for a two-dimensional example). A scatterer represents some region of a disordered medium, and it can be characterized by a $2d\times 2d$ random scattering matrix $S_{\alpha\beta}$. This model is a d-dimensional generalization of the one-dimensional model of Ref. 9. Amplitudes B_α of the waves outgoing from a scatterer are related to the amplitudes A_β of the incoming waves by

$$B_\alpha=\sum_{\beta=1}^{2d}S_{\alpha\beta}A_\beta \quad (\alpha=1,\ldots,2d). \tag{1}$$

Writing down Eqs. (1) for each scatterer, one obtains Nz coupled equations (N is the number of sites in the lattice, $z=2d$ is the coordination number) for the Nz amplitudes.[14] These equations supplemented by boundary conditions at perfect leads[12] (through which the electrical current enters and leaves the lattice) enable one, in principle, to calculate the current, and thus the conductance of the lattice. I will not discuss boundary conditions in more detail, since they will not be needed in the RG approach employed below. However, it is important to realize that the outlined discussion provides a well-defined procedure for a calculation of the conductance of a lattice of random quantum scatterers.

Now I apply a Migdal-Kadanoff type RG transformation[15-17] to this model. The lattice is partitioned into equal hypercubic cells of edge b (b

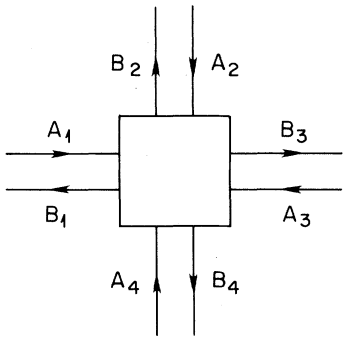

FIG. 1. A two-dimensional scatterer with four incoming (amplitudes A_α) and four outgoing (amplitudes B_α) channels.

is the scaling factor). The direction of the average current is singled out, and bonds in the other $d-1$ directions (perpendicular to the current direction) are cut.[17] Thus each cell now contains b^{d-1} chains in parallel, while each chain contains b scatterers in series.

The b scatterers along a chain can be combined according to the rules worked out in Ref. 9. The important assumption is that the phase of the reflection (or transmission) amplitudes in the S matrix of a scatterer is completely random. This assumption implies that the lattice spacing L_0 (i.e., the size of a disordered region represented by a scatterer) is much larger than some microscopic length l at which the phases are randomized. It is this assumption that enables one to combine directly the resistances, rather than the scattering amplitudes, of the scatterers.[9] For b scatterers in series the result is

$$\rho_1=(1+\rho)^b-1, \tag{2}$$

where ρ_1 and ρ are typical or scale (dimensionless) resistances of a chain and of a single scatterer, respectively. In a more rigorous treatment one should scale the whole distribution of resistances, rather than a typical resistance, as explained in Ref. 9.

To complete the renormalization procedure I now combine the resistances of the b^{d-1} chains in parallel. This is done according to the classical Ohm's law, since there is no scattering between different chains. Thus the renormalized resistance ρ' (i.e., the typical resistance of a cell of b^d scatterers) is

$$\rho'=b^{-(d-1)}[(1+\rho)^b-1], \tag{3}$$

or in infinitesimal form (i.e., $b=1+\zeta$, where $\zeta \to 0$)

$$\rho'=\rho+\zeta[(1+\rho)\ln(1+\rho)-(d-1)\rho]. \tag{4}$$

This transformation has a nontrivial fixed point ρ_c only for $d>2$, which is in agreement with the scaling theory of localization.[1] For $d-2\equiv\epsilon\ll 1$ the recursion relation (4) can be examined analytically with the following results: $\rho_c=2\epsilon$, $\nu=1/\epsilon$, and $t=(d-2)\nu=1$. Thus near two dimensions the RG transformation produces the asymptotically exact exponents. At $d=3$ numerical analysis leads to $\rho_c=3.92$ and $\nu=t=1.68$. This value of ν is compatible with the estimate of Sarker and Domany,[7] $1.25<\nu<1.75$, but much higher than the other estimates cited above and in Ref. 7.

So far the discussion has been limited to microscopically random but macroscopically homogeneous medium. Next, I discuss localization in a macroscopically inhomogeneous sample, with percolation disorder. Percolation can be incorporated into the present model by introducing a probability p that a site is occupied by a (random) scatterer. Then $1-p$ is the probability that a site is empty, which corresponds to total reflection of the incident wave. The case $p=1$ corresponds to a macroscopically homogeneous disordered medium considered above.[18] Since percolation disorder in this model occurs at scale L_0 (lattice spacing) it represents a macroscopic disorder, as opposed to the microscopic disorder at scale l discussed above.[19]

There are two parameters in this model: the occupation probability p and the typical resistance ρ of a scatterer. Thus one can expect a critical curve $\rho_c(p)$ separating between metallic and insulating regions in the p-ρ plane. Such a "phase diagram" has been recently suggested by Khmelnitskii.[20] Below, I calculate numerically the $\rho_c(p)$ curve (thick solid line in Fig. 2) for the outlined model by the RG transformation described above.

The procedure is essentially the same as for classical percolation networks.[16,17] The only difference is that resistances along a chain are now

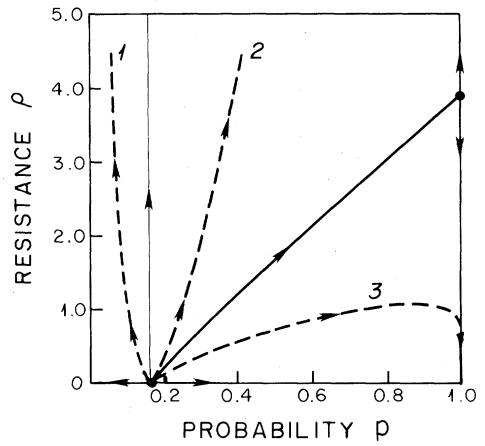

FIG. 2. Flow diagram generated by recursion equations (5) and (7), with $\zeta=0.02$, at three dimensions. The percolation threshold $p_c=0.16$, the critical resistance for macroscopically homogeneous localization $\rho_c=3.92$. The percolation and localization fixed points are (0.16; 0) and (1; 3.92), respectively. The thick solid line is the critical line $\rho_c(p)$. The three dashed lines are representative flow lines in the three regions discussed in the text. These three lines approach the three trivial fixed points $(0;\infty)$, $(1;\infty)$, and $(1;0)$, respectively.

combined according to Eq. (2), rather than according to the classical Ohm's law. Because of the approximate character of the transformation, the results depend somewhat on whether one renormalizes resistances or conductances.[17] To facilitate comparison with the procedure of Refs. 16 and 17, I renormalize the (typical) dimensionless conductance σ of a random scatterer. In the limit $b \to 1$ the two recursion relations are

$$p' = p + \zeta [p \ln p - (d-1)(1-p)\ln(1-p)], \tag{5}$$

$$\sigma' = \sigma - \zeta \{\sigma(\sigma+1)\ln(1+1/\sigma) - (d-1)\sigma[1+p^{-1}(1-p)\ln(1-p)]\}. \tag{6}$$

For $\sigma \gg 1$ results of Refs. 16 and 17 are recovered. Since the "phase diagram" is conveniently plotted and discussed in terms of $\rho = 1/\sigma$, rather than σ, I rewrite Eq. (6) as

$$\rho' = \rho + \zeta \{(1+\rho)\ln(1+\rho) - (d-1)\rho[1+p^{-1}(1-p)\ln(1-p)]\}. \tag{7}$$

The flow diagram generated by Eqs. (5) and (7), with $\zeta = 0.02$, at $d = 3$ is shown in Fig. 2. The flow line connecting the two nontrivial fixed points —percolation fixed point $(p_c, 0)$ and localization fixed point $(1, \rho_c)$—represents the critical line $\rho_c(p)$ [p_c is the percolation threshold, $\rho_c \equiv \rho_c(1)$ is the critical resistance for macroscopically homogenous localization]. This line divides the p-ρ plane into three distinct regions, as suggested by Khmelnitskii[20]: The region to the left of the $p = p_c$ line corresponds to "classical" localization, while the regions below and above the critical line correspond to extended and localized (in the Anderson sense) states, respectively. Since the critical flow line is directed from the percolation fixed point to the (macroscopically homogeneous) localization fixed point, it can be concluded that percolation disorder (for $p > p_c$) does not affect the exponents of the Anderson transition. It does, however, affect the width of the critical region. More work is needed to obtain quantitative information about this effect, as well as about crossover phenomena near the percolation fixed point.

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