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# Stationary System of Two Masses Kept Apart by Their Gravitational Spin-Spin Interaction

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An exact vacuum solution of Einstein's field equations is presented, describing two isolated bodies balanced by their gravitational spin-spin interaction.

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It is well known that Curzon's static bipolar solution of the Einstein field equations[1] which describes the axisymmetric gravitational field of two separated masses fails to satisfy the condition of elementary flatness on the part of the axis between the two masses. In physical terms this can be interpreted as meaning that the masses are held apart by a strut.[2,3] With the recently developed techniques for generating stationary axisymmetric solutions from static ones, the question arose whether it is possible to stabilize two masses by addition of angular momentum. This is indeed the case.

The metric for space-time has the usual form

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f\,(dt - \omega\, d\varphi)^2,$$

and Curzon's bipolar solution for equal masses is given by

$$f_0 = e^{2\chi} = \exp[-4mx/(x^2 - y^2)], \quad \gamma_0 = -m^2 \frac{1-y^2}{(x^2-y^2)^4}[(x^2-1)(x^4+6x^2y^2+y^4) + (x^2-y^2)^3], \quad \omega_0 = 0,$$

 

where the Weyl coordinates $(\rho, z)$ and the prolate spheroidal coordinates $(x, y)$ are mutually related by

$$\rho^2 = (x^2-1)(1-y^2), \quad z = xy.$$

$\gamma_0$ is only fixed up to an additive constant.

With this solution as seed metric we have used two Hoenselaers-Kinnersley-Xanthopoulos (HKX) rank-0 transformations (see HKX[4] for details) to derive the Ernst potential

$$\epsilon = e^{2\chi} Z/N; \quad Z = 1 + \lambda^2 B_+ B_- \frac{(1-x)^2(1-x^2)}{(x^2-y^2)^2} + i(1-x)\lambda\left[B_+ \frac{1+y}{(x-y)^2} - B_- \frac{1-y}{(x+y)^2}\right],$$

$$N(x,y) = Z(-x,-y), \quad B_\pm = \exp\left[-2m\left(\frac{1\mp xy}{(x\mp y)^2} + \frac{x}{x\pm y}\right)\right].$$

The parameters $\alpha_1$, $\alpha_2$, $u_1$, $u_2$ of the HKX transformation have been chosen as $\lambda = \alpha_1 = \alpha_2$, $u_1 = -u_2 = \frac{1}{2}$. For this Ernst potential the metric is given by

$$f = A/[A + 2(G + H\cos\tau + I\sin\tau)], \quad \omega = -A^{-1}(R + Q\cos\tau - P\sin\tau) + k, \quad e^{2\gamma} = A\exp[2(\gamma_0 - \chi)], \tag{1}$$

with the functions

$$A = e^{2\chi}\operatorname{Re}(ZN^*), \quad 2I = e^{2\chi}\operatorname{Im}(ZN^*), \quad 4H = NN^* - e^{4\chi}ZZ^*, \quad 4G = NN^* + e^{4\chi}ZZ^* - 2A,$$

and

$$R = \lambda\left\{\frac{B_+}{x-y}[(1+x)(1-y) - e^{4\chi}(1-x)(1+y)] + \frac{B_-}{x+y}[(1+x)(1+y) - e^{4\chi}(1-x)(1-y)]\right\}$$

$$- \lambda^3 B_+ B_- \frac{x^2-1}{x^2-y^2}\left\{\frac{B_+}{(x-y)^3}[(1+x)^3(1-y) - e^{4\chi}(1-x)^3(1+y)] + \frac{B_-}{(x+y)^3}[(1+x)^3(1+y) - e^{4\chi}(1-x)^3(1-y)]\right\},$$

$$Q = R(e^{4\chi} \to -e^{4\chi}),$$

$$P = 4my\frac{x^2-1}{x^2-y^2}A + 2e^{2\chi}\left\{\lambda^2(x^2-1)\left[B_+^2\frac{1-y^2}{(x-y)^3} - B_-^2\frac{1-y^2}{(x+y)^3} - 2B_+B_-\frac{y}{x^2-y^2}\right] + 2\lambda^4 B_+^2 B_-^2 y\left(\frac{x^2-1}{x^2-y^2}\right)^4\right\}.$$

Moreover, $\tau$ denotes the parameter of an Ehlers transformation, which is still available, and $k$ is a constant of integration.

For the following calculations the axis outside the two objects located at $x = \pm 1$, $y = 1$ is described by $x = \pm 1$, $y \neq 1$ whereas the inner axis is given by $x \neq \pm 1$, $y = 1$. The parameter $\tau$ of the Ehlers transformation is fixed by the choice

$$\tau = \tfrac{1}{2}\pi.$$

As a consequence $\omega$ takes the form

$$\omega = -A^{-1}(R - P) + 4\lambda,$$

where the constant $k$ has been chosen such that $\omega(y \to \infty) = 0$. This guarantees asymptotic flatness and, moveover, that the axis outside the two objects is regular which means that $\omega|_{x=\pm1, y\neq1} = 0$. On the inner axis we obtain

$$\omega|_{x\neq\pm1, y=1} = -\frac{4e^{2m}}{1-\alpha}\{\alpha^2 + \alpha[(1+m)e^{-2m} - 1] - me^{-2m}\},$$

where the rescaled parameter

$$\alpha = \lambda e^{-2m}$$

is used. Hence one condition for regularity is

$$\alpha^2 + \alpha[(1+m)e^{-2m} - 1] - me^{-2m} = 0, \tag{2}$$

which also ensures the vanishing of $g_{\varphi\varphi}$ on the axis. Elementary flatness on the axis requires

$$\lim_{\rho\to0}\frac{g_{\varphi\varphi}}{\rho^2 g_{\rho\rho}} = \lim_{\rho\to0} e^{-2\gamma}\left[1 - \left(\frac{f\omega}{\rho}\right)^2\right] = 1. \tag{3}$$

With condition (2) it can be seen that $\omega \simeq (y-1) \times \omega_1(x)$ for $y \simeq 1$ and hence $\lim_{\rho\to0} f\omega/\rho = 0$. Thus we get from (1) and (3)

$$\lim_{y\to1} e^{-2\gamma} = \exp(-2m^2)(1-\alpha^2)^{-2},$$

where we disposed of the free additive constant in $\gamma_0$ by requiring $\gamma_0(x = \pm 1) = 0$, equivalent to the rescaling $\gamma_0 + m^2 \to \gamma_0$. Demanding this expression to be 1 we obtain the second equation

$$\exp(m^2) = (1-\alpha^2)^{-1}. \tag{4}$$

It follows immediately that the real parameters $\alpha$ lie in the open interval $(0,1)$.

It remains to show that the equations (2) and (4) are consistently solvable which is equivalent with the existence of two numerical values $\alpha_z$ and [see (4)] $m_z = m(\alpha_z)$ for $\alpha$ and $m$ such that the function

$$F(\alpha) = \alpha^2 + \alpha\left(\{1 + [-\ln(1-\alpha^2)]^{1/2}\}\exp\{-2[-\ln(1-\alpha^2)]^{1/2}\} - 1\right) - [-\ln(1-\alpha^2)]^{1/2}\exp\{-2[-\ln(1-\alpha^2)]^{1/2}\}$$

possesses a zero at the point $\alpha_z \in (0,1)$. For $\alpha \ll 1$ we find that the function $F(\alpha)$ is negative because $F(\alpha \ll 1) = -\alpha + O(\alpha^2)$. On the other hand, we obtain positive values for $F(\alpha)$ for $\alpha > 0.97$ such that the existence of a zero of $F(\alpha)$ in the interval $(0,1)$ is ensured.

The two numerical values $\alpha_z$ and $m_z$ (up to four decimal places $\alpha_z = 0.9635$ and $m_z = 1.6235$) determine uniquely a one-parameter family (this parameter is the focal length of the prolate spheroidal coordinates which we have normalized to unity) of stationary, axisymmetric, asymptotically flat gravitational fields, describing two objects with positive mass $M = 2\alpha_z \exp(2m_z)$ rotating in the same direction and balanced by their gravitational spin-spin interaction.

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