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[12]We define the norm of the matrix $A$ as $\|A\| \equiv \sup_{|c| \neq 0} |Ac|/|c|$, where $c$ ranges over all vectors in the Hilbert space.

[13]A set $E$ of real numbers is said to be relatively dense if there exists a number $L < \infty$ such that any interval on the real axis of length $L$ contains at least one member of $E$.

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[17]A physical criterion for being away from resonance is to have the frequency of the periodic potential incommensurate with the spectrum of $H_0$.

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# Evidence for Universal Chaotic Behavior of a Driven Nonlinear Oscillator

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A bifurcation diagram for a driven nonlinear semiconductor oscillator is measured directly, showing successive subharmonic bifurcations to $f/32$, onset of chaos, noise band merging, and extensive noise-free windows. The overall diagram closely resembles that computed for the logistic model. Measured values of universal numbers are reported, including effects of added noise.

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Our purpose is to report detailed measurements on a driven nonlinear semiconducting oscillator and to make quantitative comparisons with the predictions of a simple model of period-doubling bifurcation as a route to chaos,[1-3] which stems from earlier work in topology.[4] There is surprising agreement, lending support to the belief and the hope that some nonlinear systems can be approximately understood by a universal model, as has been suggested by some experiments.[5,6] This upsurge of interest in nonlinear behavior has been triggered by the remarkable result that deterministic computer iterations of such a simple nonlinear recursion relation as the logistic equation

$$x_{n+1} = \lambda x_n (1 - x_n) \tag{1}$$

yield exceedingly complex pseudorandom or chaotic behavior.[2,3] The results are best summarized by a bifurcation diagram[7-9]: a scatter plot of the iterated value $\{x_n\}$ versus the control parameter $\lambda$, which shows that as $\lambda$ is increased $\{x_n\}$ displays a series of pitchfork bifurcations at $\lambda_n$, with period doubling by $2^n$, $n = 1, 2, \ldots$ . These converge geometrically, as $\lambda_c - \lambda_n \propto \delta^{-n}$, to the onset of chaos at $\lambda_c$, where $\{x_n\}$ becomes aperiodic; in the chaotic regime, $\lambda > \lambda_c$, noise bands merge and there exist narrow periodic windows in a specific order and pattern.[4] This model is quantified by universal numbers as $n \to \infty$: $\delta = 4.669\ldots$, and the pitchfork scaling parameter $\alpha = 2.502\ldots$, first computed by Feigenbaum. Other universal numbers characterize the spectral power density[10,11] and effects of noise.[8,12]

Our experimental system is a series $LRC$ circuit driven by a controlled oscillator, described by $L\ddot{q} + R\dot{q} + V_c = V_d(t) = V_0 \sin(2\pi f t)$, where $V_c$ is the voltage across a Si varactor diode (type 1N953 supplied by TRW Company), which is the nonlinear element. Under reverse voltage, $V_c = q/C$,

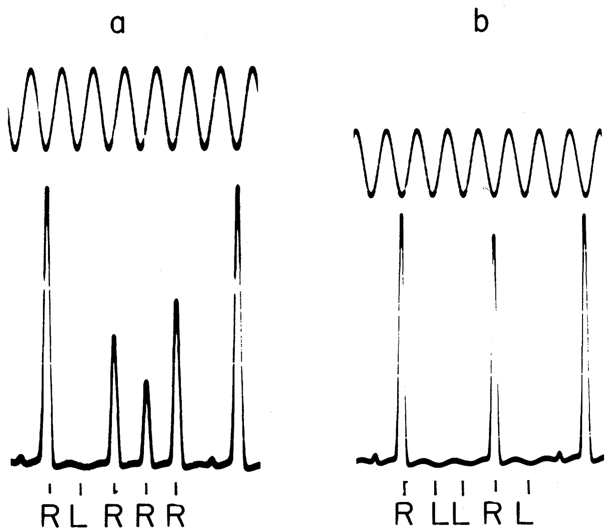


FIG. 1. (a) The varactor voltage $V_c(t)$ and the driving oscillator voltage $V_d(t)$ (upper) for period 6 window at 2.073 V; the pattern is RLRRR, and describes the sequence of visitation of the oscillator to its states according to whether it is to the right or left of zero, following the notation of Ref. 4. (b) Period 6 window at 3.338 V, with different pattern RLLRL.

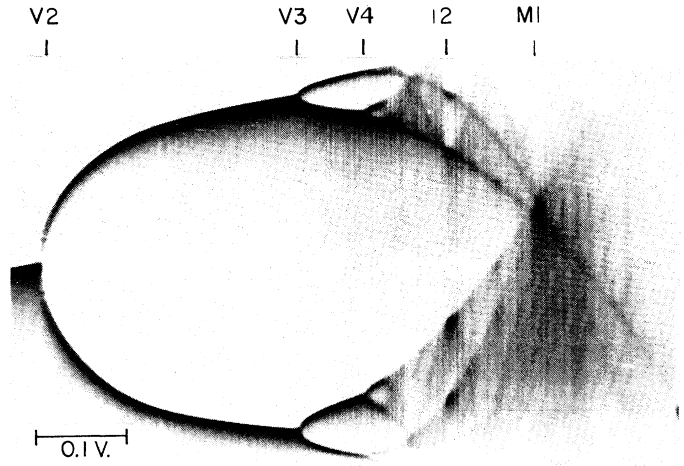


FIG. 3. Expansion of a region of Fig. 2, showing bifurcation thresholds $V_2$, $V_3$, and $V_4$; window of period 12; and band merging $M_1$.

where $C \simeq C_0/[1+V_c/0.6]^{0.5}$, $C_0 \simeq 300$ pF; under forward voltage the varactor behaves like a normal conducting diode. The coil inductance $L = 10$ mH, the resistance $R = 28\ \Omega$. At low values of $V_0$, the system behaves like a high-$Q$ resonant circuit at $f_{res} = 93$ kHz; as $V_0$ is increased, the resonant frequency shifts upward and the $Q$ is lowered. It is not our intention to solve the intractable nonlinear differential equations for this system[13] but rather to do extensive and novel measurements designed to compare its behavior as fully as possible with the simple logistic model. We fix $f$ near $f_{res}$, vary the driving voltage $V_0$, and measure the varactor voltage $V_c(t)$. We assume a correspondence between $V_0$ and $\lambda$ and between $V_c$ and $x$ of Eq. (1).

A real-time display, e.g., Fig. 1, of $V_c(t)$ and $V_0(t)$ on a dual-beam oscilloscope, with $V_0$ as a parameter, clearly revealed threshold values $V_{0n}$ for bifurcation; the bifurcation subharmonics $f/2^n$ up to $f/16$; and the pattern of visitation of the oscillator to its stable points. The data shown at two different windows in the chaotic regime, both for period-6 orbits, show different patterns, as expected.[4] During the diode con-

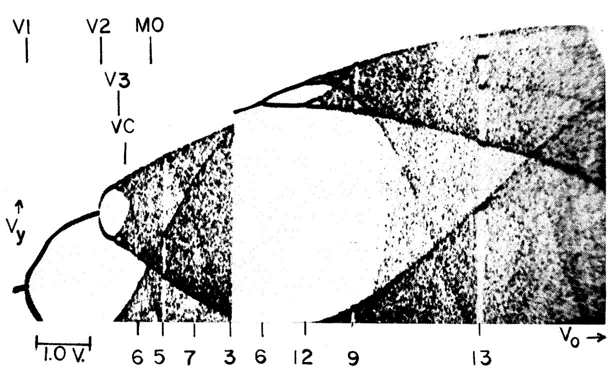


FIG. 2. Bifurcation diagram $V_y$ vs $V_0$ at $f = 96.85$ kHz, showing thresholds $V_1$, $V_2$, and $V_3$ for periods 2, 4, and 8; threshold for chaos $V_c$; band merging $M_0$; and windows of periods 6, 5, 7, 3, 6, 12, 9, and 13. The veiled lines are peaks in the spectral density in the chaotic regime.

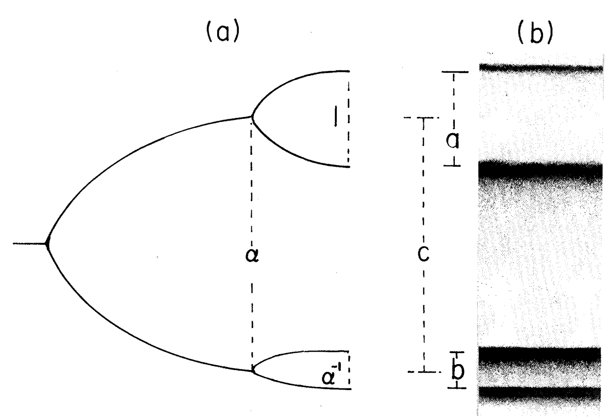


FIG. 4. (a) Schematic of universal metric scaling of pitchfork bifurcation, determined by $\alpha$ (Ref. 2). (b) Data for period 16 between $V_4$ and $V_5$, which yield the values $\alpha = a/b = 2.35$ and $\alpha = c/a = 2.61$.

TABLE I. Measured thresholds at 99 kHz.

| Period | Threshold $V_0$ rms volts | Comments |
|---|---|---|
| 2 | 0.639 | Threshold for periodic bifurcation |
| 4 | 1.567 | |
| 8 | 1.785 | |
| 16 | 1.836 | |
| 32 | 1.853 | |
| Chaos | 1.856 | Onset of noise |
| 12 | 1.901 | Window |
| 24 | 1.902 | |
| 6 | 2.073 | Window |
| 12 | 2.074 | |
| 5 | 2.353 | Window |
| 10 | 2.363 | |
| 7 | 2.693 | Window |
| 14 | 2.696 | |
| 3 | 3.081 | Wide Window |
| 6 | 3.338 | |
| 12 | 3.711 | |
| 24 | 3.821 | |
| 9 | 4.145 | Window |
| 18 | 4.154 | |

ducting half cycle, $V_c$ is compressed toward the zero line; in the reverse half cycle, $V_c$ has a set of discrete values, which correspond to the upper half of the bifurcation diagram.

To analyze $V_c$, a window comparator was constructed which selected components between $V_y$ and $V_y + \Delta V$, $\Delta V \simeq 10$ mV. A vertical scan of $V_y$ simultaneously with a slower horizontal scan of $V_0$ on an oscilloscope yielded Figs. 2 and 3, the first measured bifurcation diagram for a physical system showing subharmonic sequences. It has a striking resemblance to the computed diagram,[7,8] including bifurcation thresholds, onset of chaos, band merging, noise-free windows, and the subtle veiled structure, corresponding to regions of high probability.[8] The diagram allows a direct measurement of the number $\alpha$; from the expanded region, Fig. 4, the ratio of the pitchfork splittings is directly measured in a series of ten similar measurements:

$$\alpha = 2.41 \pm 0.1. \tag{2}$$

The diagram shows at least five noise-free windows, which bifurcate within the window: From Fig. 2 and Table I, at $V_0 = 3.081$ V, a noise-free window of period 3 appears, which bifurcates to periods 6, 12, and 24 before onset of chaos again.

The power spectral density of $V_c(t)$ was measured with a spectrum analyzer with 40 dB dynamic range, which showed the expected subharmonics $\frac{1}{2}$; $\frac{1}{4}$, $\frac{3}{4}$; $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, etc., rather symmetrically displayed about $f/2$. The data shown in Fig. 5 were obtained with a more sensitive spectrum analyzer with 85 dB of dynamic range, sensitivity of 300 nV, and range $f = 0$ to 50 kHz $\geq f/2$, thus allowing observation of spectral components 95 dB below $V_0$ at $f$. Figure 5 shows periodic subharmonics to $f/32$ at $V_0$ just below the threshold for chaos $V_{0c}$; the predicted values of the individual spectral components are shown.[14] It is predicted[10] that the average heights of the peaks for a period is $10 \log 20.963 = 13.21$ dB below the previous period; the data are consistent with this, although the region between $f/2$ and $f$ is not available for exact averaging. Spectral analysis showed other noise-free windows (60 dB above noise) at periods 12, 6, 5, 7, and 9, at thresholds listed in Table I; all show bifurcations within the window. The entire $V_0$ sequence of Table I, identified by period and pattern, is consistent with the universal $U$ sequence of Metropolis, Stein, and Stein[4] (who limit computation to period $\leq 11$). From the first four threshold voltages $V_{0n}$ we calculate the convergence rate

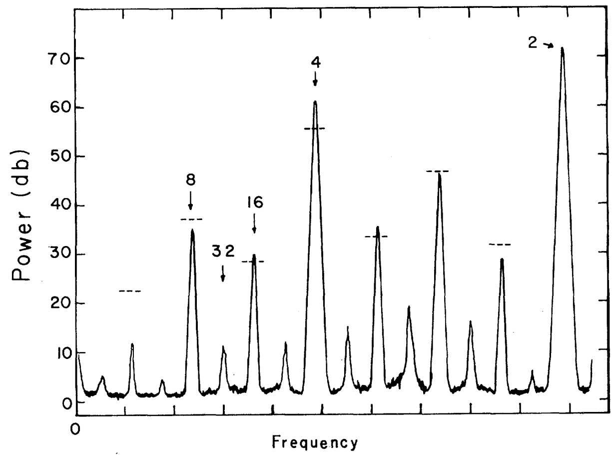


FIG. 5. Power spectral density (dB) vs frequency for $f = 98$ kHz, dynamic range 70 dB, showing subharmonics to $f/32$. The components agree with prediction (dashed bars, Ref. 14) within 2 dB rms deviation, except for the peak at $f/16$.

$$\delta_1 = \frac{V_{02} - V_{01}}{V_{03} - V_{02}} = 4.257 \pm 0.1;$$
$$\delta_2 = \frac{V_{03} - V_{02}}{V_{04} - V_{03}} = 4.275 \pm 0.1. \tag{3}$$

We observed the effect on the system of adding a random noise voltage $V_n(t)$ to $V_d(t)$. The bi-

TABLE II. Measured and predicted values for universal numbers.

| Number | | Measured | Predicted |
|---|---|---|---|
| $\delta_1$ | Eq. (3) | $4.26 \pm 0.1$ | 4.751[a] |
| $\delta_2$ | Eq. (3) | $4.28 \pm 0.1$ | 4.656[a] |
| $\delta_1$ | Period 3 window | $0.69 \pm 0.1$ | 0.979[a] |
| $\delta_2$ | Period 3 window | $3.38 \pm 0.1$ | 4.429[a] |
| $\alpha$ | | $2.41 \pm 0.1$ | 2.502[b] |
| $\epsilon$ | | $6.3 \pm 0.3$ | 6.55[c] |
| Average spectral power ratio | | 11 to 15 dB | 13.61 dB[d] |

[a]Computed from Eq. (1); cf. asymptotic limit 4.669, Ref. 2.
[b]Ref. 2.
[c]Ref. 12.
[d]Ref. 10.

furcation diagram and the power spectra were observed as $|V_n|$ was increased: periods 16, 8, 4, and 2 were successively obliterated at $V_n = 10$, 62, 400, and 2500 $\mathrm{mV_{rms}}$, respectively, yielding an average value

$$\kappa = 6.3 \tag{4}$$

for the noise voltage factor required to reduce by one the number of observable bifurcations.

To summarize, Table II compares our measured values with predicted values for some universal numbers. There is overall reasonable quantitative agreement between the data and the logistic model. The likely cause for some discrepancy in $\delta$ is that the data cannot be taken in the asymptotic limit $n \to \infty$. These are first direct measurements for $\alpha$ and $\kappa$. The strong similarity between the predicted and the observed bifurcation diagram gives further support to the utility of simple models as a key to chaotic behavior of nonlinear systems. The measurement of a bifurcation diagram is a powerful method for assessing the degree to which a particular physical system will follow this route, or other routes[14]; it is not yet known how to predict this in advance.

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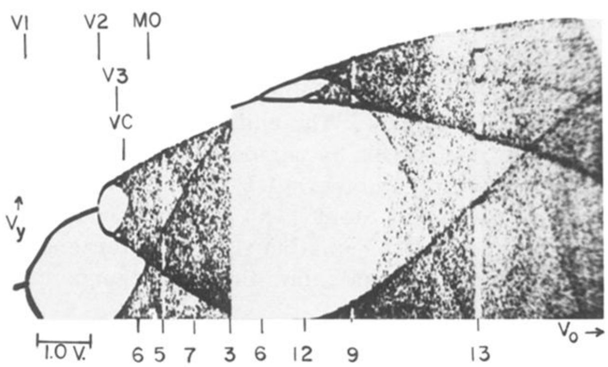


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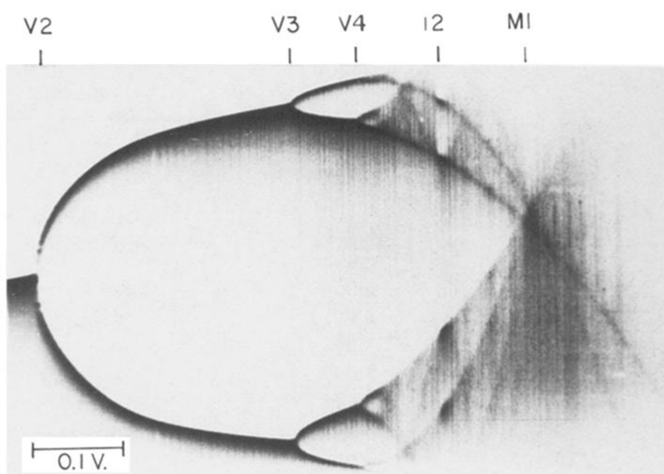


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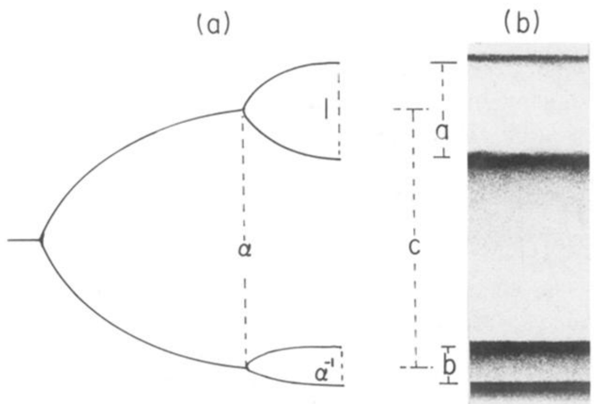


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