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Recurrence Phenomena in Quantum Dynamics

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It is proved that under any time-periodic Hamiltonian, a nonresonant, bounded quantum system will reassemble itself infinitely often in the course of time. To illustrate these results computer experiments are performed on both a pulsed quantum rotor and an electron in the field of periodic electromagnetic pulses.

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Little is known about the time evolution of quantum systems with nonstationary Hamiltonians. And yet this question is of relevance to a host of problems that range from laser photochemistry[1] to the dynamics of electrons in very small structures,[2] where perturbation treatments are seldom suitable. On a perhaps deeper level there remains the question of the behavior of quantum systems whose classical counterparts are known to display chaotic behavior, a problem which has been recognized since the early days of quantum theory.[3] Although for time-independent Hamiltonians there exist a number of theoretical results,[4-8] for nonstationary Hamiltonians we possess fewer answers. In particular, recent work by a number of authors[9-11] has indicated the possibility of quantum chaotic behavior which can be characterized by either decay of correlation functions or diffusive energy growth. Since most of these predictions are based on limited numerical calculations, it is desirable to obtain some general theoretical results which can in turn be used to both interpret and predict experimental outcomes.

This paper reports the results of such a theory and illustrates its implications for some concrete problems with numerical experiments. In particular, we prove that under any time-periodic Hamiltonian, a nonresonant, bounded quantum system (i.e., a system with a discrete quasienergy spectrum) will reassemble itself infinitely often in the course of time. This in turn implies that no strict quantum stochasticity is possible, a result which disagrees with recent predictions.[9-11] Furthermore, we perform computer experiments on both a pulsed quantum rotor and an electron in the field of periodic electromagnetic pulses, problems illustrating our rigorous results.

Consider any bounded quantum system described by a Hamiltonian H_0 that has a discrete spectrum, and subjected to a nonresonant time-periodic potential V with $V(t)=V(t+T)$ for an arbitrary period T, and such that $\|\dot{V}\|$ (Ref. 12) is bounded. We will now prove that given any initial configuration of the system, both the wave function and the energy return arbitrarily close to their initial values infinitely often. More generally, if we define an almost-periodic function, $f(t)$, to be a continuous, bounded function such that for any $\epsilon > 0$ there exists a relatively dense set $\{\tau_\epsilon\}$[13] and for *each* τ_ϵ in the set we have $|f(t+\tau_\epsilon) - f(t)| < \epsilon$ for all t, our theorem states that both the normed wave function and the energy are almost-periodic functions of time.

We start by proving almost-periodicity of the wave function. Consider the time-dependent Schrödinger equation $i\hbar\partial\Psi/\partial t = [H_0 + V(t)]\Psi$. In an expansion of the wave function, Ψ, in terms of the complete orthonormal set of eigenstates of H_0, $\{u_m(r)\}$, as $\Psi(r,t) = \sum_{m=1}^\infty a_m(t) u_m(r)$, the coefficients $a_m(t)$ make up a vector $a(t)$ which satisfies

$$i\hbar\dot{a}(t) = H(t)a(t) \tag{1}$$

and we can write

$$\|\Psi(t+\tau) - \Psi(t)\|^2 = |a(t+\tau) - a(t)|^2, \tag{2}$$

where we have defined $\|\Psi(t)\|^2 \equiv \int dr |\Psi(r,t)|^2$. Furthermore, if $H(t) = H(t+T)$ the wave function satisfies a Floquet theorem, i.e., $a(t)$ is of the form

$$a(t) = \sum_{k=1}^\infty \alpha_k \exp[iE_k t/\hbar]\Phi_k(t), \tag{3}$$

with $\Phi_k(t+T) = \Phi_k(t)$, $\Phi_k^\dagger(t)\Phi_{k'}(t) = \delta_{kk'}$ for all t.[14] The set $\{E_k\}$ is called the quasienergy spectrum. If we write α_k as $\alpha_k = r_k \exp(i\varphi_k)$ with r_k and φ_k real, it follows from Eq. (3) that

$$|a(t+NT) - a(t)|^2 = 2\sum_{k=1}^\infty r_k^2 \left(1 - \cos\frac{E_k NT}{\hbar}\right) \tag{4}$$

for any integer N. Since the wave function is normalized, we have $\sum_{k=1}^\infty r_k^2 = |a(t)|^2 = \|\Psi(t)\|^2 = 1$, an equality which implies that given $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that $\sum_{k=n+1}^\infty r_k^2 < \epsilon/8$. We can then write the following inequality:

$$\sum_{k=n+1}^\infty r_k^2 \left(1 - \cos\frac{E_k NT}{\hbar}\right) \leq 2\sum_{k=n+1}^\infty r_k^2 < \epsilon/4. \tag{5}$$

We next consider the function $f(x) = \sum_{k=1}^n [1 - \cos(E_k xT/\hbar)] \geq 0$. By our definition of nonresonance the eigenvalues E_k are discrete so that this is a finite sum of periodic functions, and therefore for any $\delta > 0$, the set of integers $\{N_\delta\}$ such that $|f(x+N_\delta) - f(x)| < \delta$ for all x is relatively dense.[15] In particular, for $\delta = \epsilon/4$ and $x = 0$ there exists a relatively dense set of integers $\{N\}$ such that $f(N) < \epsilon/4$ and since each $r_k \leq 1$, we have

$$\sum_{k=1}^n r_k^2 \left(1 - \cos\frac{E_k NT}{\hbar}\right) < \epsilon/4.$$

Combining this result with Eqs. (5), (4), and (2), we obtain

$$\|\Psi(t+NT) - \Psi(t)\|^2 < \epsilon \tag{6}$$

for all times t and for a relatively dense set of times $\{NT\}$.

Having proved the reassembly process for the wave function, we now proceed to prove that the energy will also recur infinitely often with arbitrary accuracy. The time evolution of the energy is determined by $\dot{E}(t) = \langle\Psi|\dot{H}(t)|\Psi\rangle = a^\dagger \dot{V} a$, and so a simple integration gives

$$E(t) = E_0 + \int_0^t a^\dagger(t')\dot{V}(t')a(t')dt'. \tag{7}$$

From the first part of our theorem, we know that the vector $a^\dagger(t)$ is almost-periodic and that $\dot{V}(t) = \dot{V}(t+T)$. Since in addition we have assumed $\|\dot{V}\|$ to be bounded, $\dot{V}(t)a(t)$ is an almost-periodic vector. Therefore the integrand of Eq. (7), being the product of two bounded almost-periodic vectors, is itself an almost-periodic scalar. Because we have excluded resonant growth of the energy, and because the integral of an almost-periodic function is almost-periodic if bounded,[15] it follows that the energy is an almost-periodic function, i.e.,

$$|E(t+\tau) - E(t)| < \epsilon \tag{8}$$

for all times t with $\{\tau\}$ a relatively dense set. This completes the proof of our theorem.[16]

In order to illustrate these results, we studied two different quantum systems. The first is the quantum version of a classical nonintegrable system which exhibits large-scale stochasticity for some values of a control parameter and which has been claimed to exhibit diffusionlike behavior for the energy in the quantum limit.[9] The Hamiltonian for this periodically kicked rotor is

$$H = P_\theta^2/2I - \omega_0^2 I \cos\theta \sum_{n=-\infty}^\infty \delta(t/T - n), \tag{9}$$

where θ is the angle, P_θ is the angular momentum, I is the moment of inertia, and the delta functions are understood to be the limit of very narrow Gaussian pulses.

Following the technique of Casati *et al.*[9] we expand Ψ in terms of the eigenstates of $H_0 = P_\theta^2/2I$ as

$$\Psi(\theta,t) = (2\pi)^{-1}\sum_{n=-\infty}^\infty a_n(t)e^{in\theta}$$

to obtain the map

$$a_n(t+T^+) = \sum_{r=-\infty}^\infty a_r(t) b_{n-r}(k)\exp(-ir^2\tau/2), \tag{10}$$

where $k=\omega_0^2IT/\hbar$, $\tau=\hbar T/I$, and $b_s(k)=i^sJ_s(k)$ with J_s the ordinary Bessel function of the first kind and order s. Using this map we have computed the energy $E(t)=\sum_{n=-\infty}^{\infty}(n^2/2I)\hbar^2|a_n(t)|^2$ for several values of k and τ, checking the normalization condition $\sum|a_n|^2=1$ to 16 digits at every iteration. A typical result for $E(t)$ is shown in Fig. 1, where we show its time evolution in time units of number of pulses for the case $k=2.871$, $\tau=2.532$ and with the initial configuration in the ground state. As can be seen, the excursions in energy are not only bounded but also recur many times.

The second problem that we studied corresponds to the dynamics of a bounded electron under the influence of a periodic string of electromagnetic pulses. This question, which is relevant to the behavior of electrons in very small structures acted upon by either the electromagnetic pulses of nearby switching elements or laser radiation, has complicated dynamics in the classical limit. Consider an electron in an infinite square-well potential which is acted upon by a set of electromagnetic pulses of strength ϵ. The Hamiltonian of the system is then given by

$$H=p^2/2m-e\epsilon x\sum_{n=-\infty}^{\infty}\delta(t/T-n), \tag{11}$$

with m the mass of the electron, e its charge, p its linear momentum, x its displacement, ϵ the pulse strength, and T the time between pulses. As in the case of the quantum rotor, this problem can be shown to satisfy the requirements of our theorem since the norm of the dipole matrix is finite.

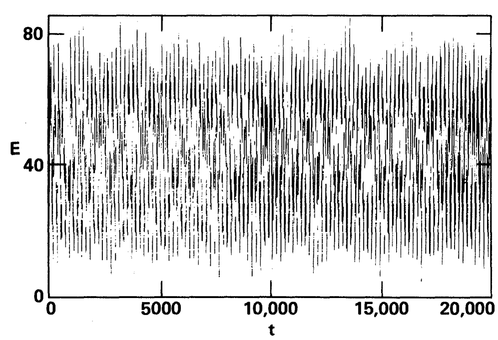

FIG. 1. The expectation value of the energy in units of $\hbar^2/I$ as a function of time in units of the number of pulses applied to the quantum rotor. The initial configuration is the ground state; $k=2.871$ and $\tau=2.532$. A total of 201 states were used, and normalization was checked to within 10^{-16}.

Proceeding as before, we constructed a quantum map from which we computed $E(t)$ as a function of both ϵ and T for thousands of pulses. Although a detailed account of our results will be published elsewhere, the point to be stressed is that the energy behaved very much as in the case of Fig. 1; it both was bounded and recurred many times.

The results we have just presented are relevant to a large variety of quantum systems beyond those explicitly treated here. In particular, we should stress that, although for the actual construction of the mappings discussed above we used a single-electron Hamiltonian, our theorem also applies to any bounded many-body problem.[17] This in turn implies that predictions can be made ranging from the behavior of molecules in the presence of monochromatic radiation to the response of very small metallic particles and electrons in superlattice structures.[18] Furthermore, we should mention that only part of the difference between classical dynamical systems exhibiting mixing behavior and the quantum analog can be ascribed to the fundamental limitation set by the finite value of Planck's constant on the decay of the correlation function. Although limiting the complexity that can develop in the wave function, this mechanism itself does not quarantee that the system will reassemble itself infinitely often. More is required, as we have shown here.

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[1]See, for instance, special issue on laser chemistry, Physics Today 33, No. 11, 25–59 (1980); A. Ben Shaul, Y. Haas, K. L. Kompa, and R. D. Levine, *Lasers and Chemical Change* (Springer, Berlin, 1981).

[2]For an extensive collection of articles, see *VLSI Electronics, Microstructure Science*, edited by N. G. Einspruch (Academic, New York, 1981), Vols. 1 and 2.

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[12]We define the norm of the matrix A as $\|A\| \equiv \sup_{|c|\neq 0}|Ac|/|c|$, where c ranges over all vectors in the Hilbert space.

[13]A set E of real numbers is said to be relatively dense if there exists a number $L<\infty$ such that any interval on the real axis of length L contains at least one member of E.

[14]This particular result, which has been previously stated by F. Gesztesy and H. Mitter, J. Phys. A 14, L79 (1981), applies to any periodic Hermitian operator.

[15]A. S. Besicovich, *Almost Periodic Functions* (Cambridge Univ. Press, Cambridge, England, 1932).

[16]We should point out that recurrence in Ψ does not necessarily imply recurrence in E, since the overall envelope of the wave function could reassemble itself with enough small-scale structure so as to produce a large change in energy.

[17]A physical criterion for being away from resonance is to have the frequency of the periodic potential incommensurate with the spectrum of H_0.

[18]See, for example, R. Dingle, in *Advances in Solid State Physics*, edited by H. J. Queisser (Pergamon, New York, 1975).

Evidence for Universal Chaotic Behavior of a Driven Nonlinear Oscillator

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A bifurcation diagram for a driven nonlinear semiconductor oscillator is measured directly, showing successive subharmonic bifurcations to $f/32$, onset of chaos, noise band merging, and extensive noise-free windows. The overall diagram closely resembles that computed for the logistic model. Measured values of universal numbers are reported, including effects of added noise.

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Our purpose is to report detailed measurements on a driven nonlinear semiconducting oscillator and to make quantitative comparisons with the predictions of a simple model of period-doubling bifurcation as a route to chaos,[1-3] which stems from earlier work in topology.[4] There is surprising agreement, lending support to the belief and the hope that some nonlinear systems can be approximately understood by a universal model, as has been suggested by some experiments.[5,6] This upsurge of interest in nonlinear behavior has been triggered by the remarkable result that deterministic computer iterations of such a simple nonlinear recursion relation as the logistic equation

$$x_{n+1}=\lambda x_n(1-x_n) \qquad (1)$$

yield exceedingly complex pseudorandom or chaotic behavior.[2,3] The results are best summarized by a bifurcation diagram[7-9]: a scatter plot of the iterated value $\{x_n\}$ versus the control parameter λ, which shows that as λ is increased $\{x_n\}$ displays a series of pitchfork bifurcations at λ_n, with period doubling by 2^n, $n=1,2,\ldots$. These converge geometrically, as $\lambda_c-\lambda_n \propto \delta^{-n}$, to the onset of chaos at λ_c, where $\{x_n\}$ becomes aperiodic; in the chaotic regime, $\lambda>\lambda_c$, noise bands merge and there exist narrow periodic windows in a specific order and pattern.[4] This model is quantified by universal numbers as $n\to\infty$: $\delta=4.669\ldots$, and the pitchfork scaling parameter $\alpha=2.502\ldots$, first computed by Feigenbaum. Other universal numbers characterize the spectral power density[10,11] and effects of noise.[8,12]

Our experimental system is a series *LRC* circuit driven by a controlled oscillator, described by $L\ddot{q}+R\dot{q}+V_c=V_d(t)=V_0\sin(2\pi f t)$, where V_c is the voltage across a Si varactor diode (type 1N953 supplied by TRW Company), which is the nonlinear element. Under reverse voltage, $V_c=q/C$,

