# Onset of Diffusion and Universal Scaling in Chaotic Systems

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A new type of universality associated with the onset of a *deterministic* diffusion for systems described by iterative one-dimensional maps is reported. The diffusion coefficient $D$ plays the role of an order parameter with a universal critical exponent. For the presence of external noise, the existence of a universal scaling function $d$ is shown. An analytic expression is derived for $d$ which is in good agreement with results of a numerical experiment.

A central issue in the study of turbulence and chaos is the routes which lead into the chaotic state. A particular question is whether there are universality classes describing the onset of chaos similar to the theory of critical phenomena. Such questions are studied conveniently in models of iterative one-dimensional (1D) maps where two types of universality have been discovered. The first type was found by Feigenbaum for the period-doubling onset of chaos.[1] Critical properties have been studied in the absence[2] and in the presence of external noise[3-5] in which case the existence of a universal scaling function was shown. A simpler type was found by Pomeau and Manneville for the onset of intermittent chaos[6,7] where the universal effects of external noise have also been investigated.[7,8] In both cases the overall shape of the maps does not influence the critical exponents. The universality classes are determined only by the limiting behavior of the maps close to a particular point.

The purpose of this Letter is to report a new type of universality. We have found that certain 1D maps can produce a diffusive motion which sets in at a critical value $\mu_c$ of a parameter $\mu$. This diffusion is fundamentally different from conventional diffusion as it is self-generated by a purely *deterministic* system. It occurs even in the absence of external random forces. The diffusion coefficient $D$ plays the role of an order parameter. Its critical exponent is $1/\alpha$, where $\alpha$ characterizes the map close to its maximum. The universality classes only depend on the type of maximum; global details are irrelevant. When external noise of strength $\sigma$ is added we find universal scaling properties. At $\mu_c$ the diffusion coefficient behaves like $\sigma^{1/\alpha}$. We show the existence of a universal scaling function $d(z)$ describing the critical behavior such that $D = \sigma^{1/\alpha} d((\mu - \mu_c)/\sigma)$. An analytic expression is derived for $d(z)$ which agrees well with the re-

sults of a numerical experiment. The noise strength $\sigma$ is thus a scaling variable which can be seen in analogy to the external field in magnetic phase transitions. Although the critical exponents are simple (as is the case for intermittent chaos[7]) this does not imply a mean-field behavior. These results are obtained by deriving a discrete master equation and calculating the diffusion coefficient. Such deterministic diffusive motions play a role in driven Josephson junctions[9,10] and other physical systems.[11] It can be shown that under certain approximations the differential equations for the Josephson junction reduce to a 1D map.[12] This map is contained in the universality classes [Eqs. (3)] which we study in this paper.

The diffusive motion is illustrated in Fig. 1 by numerical results for a particular example. Consider the discrete dynamical system

$$x_{t+1} = x_t - \mu \sin(2\pi x_t). \tag{1}$$

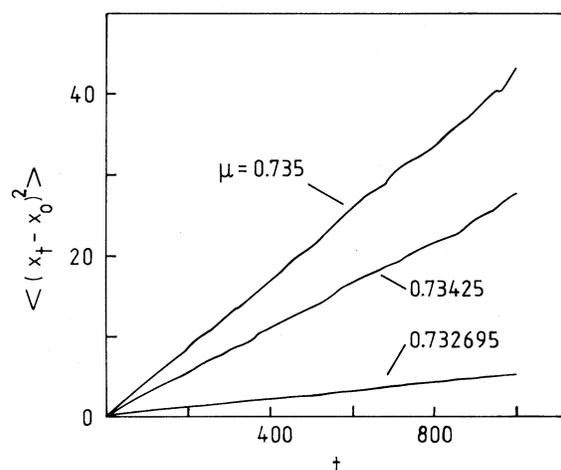Below a critical parameter value $\mu_c = 0.732\,644\,13$



FIG. 1. Mean-square displacements as a function of time for the dynamical system Eq. (1).

the system exhibits period doubling and chaotic solutions $x_t$ of the usual type. For $\mu > \mu_c$ we find a diffusive motion as is demonstrated by the mean-square displacements $\langle (x_t - x_0)^2 \rangle$ (Fig. 1). Here the average is taken over 2000 values of $x_0$ with $-\frac{1}{2} < x_0 < \frac{1}{2}$. The curves are straight lines as expected for a diffusion process and their slopes are given by $2D$.

Consider now more general maps

$$x_{t+1} = f(x_t) + \sigma \xi_t , \qquad (2)$$

where $\xi_t$ is a random variable with unit standard deviation and zero mean, and $\sigma \ll 1$ measures the strength of the external noise. The theory can be carried out for the purely deterministic case ($\sigma = 0$). For the sake of conciseness, however, we will formulate it immediately for the more general case including additive external noise.[13] For specific calculations we assume a Gaussian distribution $v(\xi)$ for $\xi$. For other choices of distribution the explicit results will change but not the critical exponents.[14] The iterative map $f(x)$ may be any function belonging to the following universality classes: $f(x)$ is odd and $f(x) - x$ is periodic (we assume period 1), i.e.,

$$f(-x) = -f(x) \quad \text{and} \quad f(x+n) = n + f(x). \qquad (3a)$$

$f(x)$ has a relative maximum per period located at $x_c + n$ with $-\frac{1}{2} < x_c < 0$. The limiting behavior near $x_c$ is

$$f(x) = a(\mu) - b(\mu)|x - x_c|^\alpha , \qquad (3b)$$

where the number $\alpha > 0$ characterizes the type of maximum and distinguishes the universality classes. Only this limiting behavior is relevant; otherwise the function may be arbitrary. $a(\mu)$ and $b(\mu)$ are parameters depending on a single parameter $\mu$. For the external-noise case only ($\sigma \neq 0$), we make an additional requirement:

$$|\tfrac{1}{2} - f(x)| \ll 1 - |x - x_c| \ll 1. \qquad (3c)$$

Let $\rho_t(x)dx$ denote the probability to find a value between $x$ and $x + dx$ at time $t$ if the initial value was at an arbitrary $x_0$ with $|x_0| < \frac{1}{2}$. For the step from $t$ to $t+1$ conservation of probability requires

$$\rho_{t+1}(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_t(x) v(\xi) \delta[y - f(x) - \sigma \xi] \, dx \, d\xi. \qquad (4)$$

Introducing unit cells centered at $x = l$ we define

$$p_t(l) = \int_{l-1/2}^{l+1/2} \rho_t(y) \, dy , \qquad (5)$$

the conditional probability to find a value in the $l$th cell at time $t$ if the initial value was in the 0th cell. From Eq. (4) we obtain

$$p_{t+1}(l) = \int_{-\infty}^{\infty} \int_{r(x)}^{s(x)} \rho_t(x) v(\xi) \, d\xi \, dx \qquad (6)$$

with $r(x) = [l - \frac{1}{2} - f(x)]/\sigma$ and $s(x) = [l + \frac{1}{2} - f(x)]/\sigma$. From now on we consider the critical region $|\mu - \mu_c| \ll \mu_c$ and $\sigma \ll 1$.[15] With the neglect of terms of order $\sigma \exp(-1/2\sigma^2)$ the integrals over $d\xi$ can be carried out, leading to

$$p_{t+1}(l) = p_t(l) - \tfrac{1}{2} \int_{l-1/2}^{l+1/2} \rho_t(x) \{ \text{erfc}[- 2^{-1/2} r(x)] + \text{erfc}[2^{-1/2} s(x)] \} dx$$

$$+ \tfrac{1}{2} \int_{l+1/2}^{l+3/2} \rho_t(x) \text{erfc}[- 2^{-1/2} s(x)] dx + \tfrac{1}{2} \int_{l-3/2}^{l-1/2} \rho_t(x) \text{erfc}[2^{-1/2} r(x)] dx. \qquad (7)$$

Here erfc denotes the complementary error function.[14] For the diffusion coefficient only the long-time behavior of $p_t(l)$ is important. Note that the relevant time scale (residence time in a unit cell) can be made arbitrarily long by letting $D \to 0$ (i.e., $\mu \to \mu_c$ and $\sigma \to 0$, the case of interest). Now we can assume that for sufficiently long time the ratio $\rho_t(x)/p_t(l)$ (with $l - \frac{1}{2} \leq x \leq l + \frac{1}{2}$) approaches a distribution independent of $t$ and $l$ which we denote by $\rho(x)$.[16] For long times Eq. (7) then becomes

$$p_{t+1}(l) - p_t(l) = - \tfrac{1}{2} p_t(l) \int_{l-1/2}^{l+1/2} \rho(x) \{ \text{erfc}[- 2^{-1/2} r(x)] + \text{erfc}[2^{-1/2} s(x)] \} dx$$

$$+ \tfrac{1}{2} p_t(l+1) \int_{l+1/2}^{l+3/2} \rho(x) \text{erfc}[- 2^{-1/2} s(x)] dx + \tfrac{1}{2} p_t(l-1) \int_{l-3/2}^{l-1/2} \rho(x) \text{erfc}[2^{-1/2} r(x)] dx. \qquad (8)$$

This equation is a discrete analog of a master equation. It was solved utilizing the symmetry properties of Eq. (3a),

$$p_t(l) = N^{-1} \sum_k \{ 1 - (1 - \cos k) \int_{-1/2}^{1/2} \rho(x) \text{erfc}(2^{-1/2}[\tfrac{1}{2} - f(x)]/\sigma) dx \}^t e^{ikl} , \qquad (9)$$

where the sum is over $N$ values of $k$ in the first Brillouin zone. This result can be used to calculate

the mean-square displacement $\langle (x_t - x_0)^2 \rangle$ for long times from which the diffusion coefficient follows as

$$D = \tfrac{1}{2} \int_{-1/2}^{1/2} \rho(x) \operatorname{erfc}\{2^{-1/2}[\tfrac{1}{2} - f(x)]/\sigma\} \, dx. \tag{10}$$

Close to $\mu_c = a^{-1}(\tfrac{1}{2})$ we can write $a(\mu) = \tfrac{1}{2} + a'(\mu - \mu_c)$ and $b = b(\mu_c) = \text{const.}$ Using Eq. (3b) and neglecting terms of order $[\sigma^2 + \sigma(\mu - \mu_c)]^{1/\alpha}$ we obtain

$$D = \sigma^{1/\alpha} d((\mu - \mu_c)/\sigma), \tag{11}$$

$$d(z) = \alpha^{-1}(\sqrt{2}/b)^{1/\alpha} \rho(x_c) \int_{-2^{-1/2}a'z}^{\infty} (u + 2^{-1/2}a'z)^{-1+1/\alpha} \operatorname{erfc}(u) \, du. \tag{12}$$

This is our most general result. $d$ is a universal scaling function as it depends only on the ratio $z = (\mu - \mu_c)/\sigma$ and no longer on $\mu - \mu_c$ and $\sigma$ individually. This situation is analogous to the one, e.g., in magnetic phase transitions. $\mu - \mu_c$ corresponds to the temperature and $\sigma$ to the magnetic field. Before we discuss $d(z)$ in more detail we determine the critical exponents. For $\mu = \mu_c$ the critical exponent of the noise follows from Eq. (11), which becomes $D = \sigma^{1/\alpha} d(0)$. In the absence of noise $(\sigma \to 0)$ Eq. (10) becomes

$$D = 2\rho(x_c)(a'/b)^{1/\alpha}(\mu - \mu_c)^{1/\alpha} + O(\Delta \mu^{3/\alpha}). \tag{13}$$

The critical exponent of the order parameter is thus $1/\alpha$.
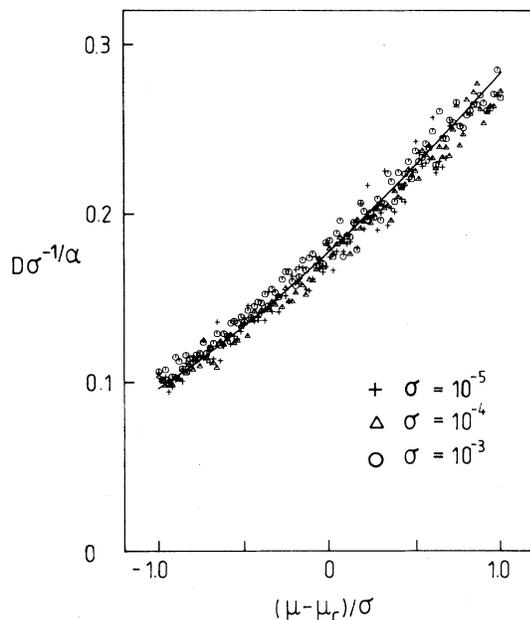


FIG. 2. Results of a computer experiment and comparison with theory. The diffusion coefficient was measured for the map $f(x) = \mu 3^{3/2}(2x^3 - x/2)$ (if $|x| < \tfrac{1}{2}$) for three different noise strengths $\sigma$ and scaled by $\sigma^{-1/\alpha} = \sigma^{-1/2}$. The data fall on a single universal curve which agrees well with the analytic universal scaling function Eq. (12) (full line).

The universal scaling function $d(\Delta \mu/\sigma)$ describes the $\mu$ and $\sigma$ dependence in the critical region for any dynamical system belonging to the universality classes Eqs. (3). Equation (12) is an analytic expression for $d(z)$ which can be computed easily. In order to illustrate our predictions and to test the quality of the analytic scaling function we have carried out a computer experiment (Fig. 2). We have measured the diffusion coefficient for three different noise strengths $\sigma$ and 100 values of $\Delta \mu/\sigma$ between $-1$ and $+1$. When $D$ is scaled to $D\sigma^{-1/\alpha}$ and $\mu - \mu_c$ is scaled to $(\mu - \mu_c)/\sigma$ (Fig. 2) the data for the three noise strengths $\sigma$ form a single universal curve as predicted by Eq. (11). The line shown in Fig. 2 is the analytic expression for the scaling function Eq. (12) which agrees well with the experimental results. Note that this agreement is achieved without any adjustable parameters (which do not occur in this theory).

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[13]Multiplicative external noise can be accounted for by substituting $\sigma = \sigma' g(x_t)$, where $g(x) \neq 0$ is continuous, bounded, and periodic. With these conventions Eq. (10) remains unchanged and in Eqs. (11) and (12) one must replace $\sigma$ by $\sigma'$ and erfc$(u)$ by erfc$(u/g(x_c))$.

[14]For a general continuous symmetric distribution $v(\xi)$ (for which we must require the existence of the second moment) erfc$(u)$ is replaced everywhere by $e(u) = 2\int v(\xi)d\xi$ integrated from $\sqrt{2}u$ to $\infty$.

[15]Since our main interest is the universal critical behavior it is sufficient to consider $|\mu - \mu_c| \ll \mu_c$ and $\sigma \ll 1$. Outside the critical region deviations from universality and scaling are expected as in equilibrium phase transitions.

[16]This assumption must be verified for each system in question. For Eq. (1) using a theorem of A. Lasota and J. A. Yorke [Trans. Am. Math. Soc. 186, 481 (1973)] one can show (T. Geisel and J. Nierwetberg, to be published) that a distribution is approached which is approximately $\rho(x) = \pi^{-1}(\frac{1}{4} - x^2)^{-1/2}$ (for $|x| < \frac{1}{2}$). The assumption may also be justified by numerical experiments (Fig. 2).