Instability Leading to Periodic and Chaotic Self-Pulsations in a Bistable Optical Cavity

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It is shown that the transmitted light from a bistable optical cavity exhibits instability leading to periodic and chaotic self-pulsations, when the relaxation time of the medium is much longer than the delay time of the feedback of light. This instability, which is different from the delay-induced one predicted recently by the authors and observed by Gibbs and co-workers, is interpreted as a self-induced Rabi nutation of the electric field vector.

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It was pointed out by several authors that the stationary response of a bistable optical system[1] becomes unstable.[2-4] Recently the present authors investigated the dynamics of a ring cavity containing a nonlinear dielectric medium and predicted that instability in such a cavity gives rise to a turbulent behavior in the transmitted light, when the incident light is sufficiently intense.[3,5] Subsequently Gibbs and others succeeded in the first experimental observation of such a phenomenon using a hybrid optical bistable device.[6]

This instability has its origin in *delayed* feedback of the light transmitted from the nonlinear medium and takes place only if the following conditions are satisfied: (1) The relaxation time of the medium γ^{-1} is shorter than the delay time of the feedback t_R, and (2) the phase shift of the electric field across the medium φ is much larger than unity.

In this Letter we investigate another limiting regime, i.e., the regime characterized by $t_R\gamma \ll 1$ and $\varphi \ll 1$, and show that a new kind of instability different from the delay-induced one occurs in this regime, leading to periodic and chaotic self-pulsations of the transmitted light under suitable conditions.

Let us start with the differential-difference equations

$$E(t) = A + BE(t-t_R)\exp\{i[\varphi(t)-\varphi_0]\}, \qquad (1a)$$

$$\gamma^{-1}\dot{\varphi}(t) = -\varphi(t) + \mathrm{sgn}(n_2)|E(t-t_R)|^2, \qquad (1b)$$

which describe the dynamics of a ring cavity containing a dispersive nonlinear medium.[5] Here $E(t)$ is the slowly varying (complex) envelope of the electric field scaled in a dimensionless form by $\{k|n_2|(1-e^{-\alpha l})/\alpha\}^{1/2}$ $(\equiv E_0)$, where k is the wave number of the incident field, n_2 the quadratic coefficient of the nonlinear refractive index, α the absorption coefficient, and l the length of the medium. The variable $\varphi(t)$ is the phase shift of the electric field across the medium, and φ_0 is a mistuning parameter of the cavity. The parameter A is the measure of the incident field E_I and is defined by $A \equiv (1-R)^{1/2}|E_I|/E_0$, where R is the reflectivity of the mirrors.[5] The parameter B, defined by $B \equiv Re^{-\alpha l/2}$ (<1), characterizes the dissipation of the electric field in the cavity.[7]

We consider here the case where $t_R\gamma \ll 1$ is satisfied and the medium is so thin that the phase shift and the dissipation are sufficiently small. More explicitly, we assume that $\varphi(t)$, φ_0, and $1-B$ are small quantities of the order of $t_R\gamma$ $(\equiv \tau_R)$. Under these assumptions, which may be regarded as mean-field assumptions,[8] $E(t)$ and A must be of the order of $\tau_R^{1/2}$ and $\tau_R^{3/2}$, respectively, and Eqs. (1a) and (1b) are approximated by a set of ordinary differential equations

$$\dot{\xi}(\tau) = a - b\xi(\tau) + i[\eta(\tau)-\eta_0]\xi(\tau), \qquad (2a)$$

$$\dot{\eta}(\tau) = -\eta(\tau) + |\xi(\tau)|^2. \qquad (2b)$$

Here, new variables ξ and η as functions of dimensionless time $\tau \equiv t\gamma$ have been introduced by $E(t) \equiv \tau_R^{1/2}\xi(\tau)$ and $\varphi(t) \equiv \tau_R\eta(\tau)$. Parameters a, b, and η_0 are related to A, B, and φ_0, respectively, through $A \equiv \tau_R^{3/2}a$, $1-B \equiv \tau_R b$, and $\varphi_0 \equiv \tau_R\eta_0$. All these new variables and parameters have been chosen to be of the order of unity.

The stationary solution of Eqs. (2a) and (2b), denoted by ξ_s and η_s, satisfies the relations $|\xi_s|^2[b^2+(|\xi_s|^2-\eta_0)^2] = a^2$ and $|\xi_s|^2 = \eta_s$. The former relation shows that, if $\eta_0 > \sqrt{3}b$, there exists a range of a in which $|\xi_s|^2$ is a threefold function of a^2 (Fig. 1). Usually, ξ_s is called *bistable* in this range, although this is not always the case.

Equation (2a) has the same form as the optical Bloch equation, if Reξ and Imξ are regarded as two components of a Bloch vector. The total phase shift $\eta(\tau)-\eta_0$ plays the role of Rabi nutation frequency and b that of damping. A characteristic of our system is that ξ exhibits a self-induced Rabi nutation as a result of the interplay between ξ and η. To see this simply, let us linearize Eqs. (2a) and (2b) with respect to $\delta\xi$ and $\delta\eta$, small deviations of ξ and η from their stationary values, as follows:

$$\delta\dot{\xi} = -b\delta\xi + i\xi_s\delta\eta + i(\eta_s-\eta_0)\delta\xi, \qquad (3a)$$

$$\delta\dot{\eta} = -\delta\eta + \xi_s\delta\xi^* + \xi_s^*\delta\xi. \qquad (3b)$$

Assume that $\eta_s \gg 1$; then $\delta\xi$ oscillates rapidly with frequency η_s. This is a Rabi nutation. This oscillation in turn modulates η through Eq. (3b). Consequently, $\delta\eta$ also oscillates with the same frequency, following $\delta\xi$ adiabatically. Thus the relation $\delta\eta \simeq (-i\xi_s^*/\eta_s)\delta\xi$ is obtained. Using this relation in Eq. (3a), we find that the damping constant for ξ is $b-1$. Therefore, if the relaxation time of the medium is shorter than the lifetime of the cavity, i.e., $b<1$, the stationary solution is unstable and the Rabi nutation is self-induced.

Detailed analysis of the linear stability reveals that the lower branch of the $|\xi_s|^2$ vs a^2 relation (Fig. 1) is always stable, while the upper branch becomes unstable if

$$(b-1)(|\xi_s|^2-\eta_0)^2-(|\xi_s|^2-\eta_0)\eta_0+b(b+1)^2>0 \qquad (4)$$

is satisfied. For $|\xi_s|\gg 1$, this reduces to $b<1$.

In the case where $a\gg 1$ and $\eta_0 \gg 1$, the explicit form of the stationary nutation is obtained beyond the linear theory as follows:

$$\xi(\tau) = ia^{1/3}b^{1/3} + [a^{2/3}(b^{-1/3}-b^{2/3})+\eta_0]^{1/2}\exp(ia^{2/3}b^{-1/3}\tau), \qquad (5a)$$

$$\eta(\tau) = a^{2/3}b^{-1/3}+\eta_0. \qquad (5b)$$

Note that the fundamental frequency of the self-pulsation $a^{2/3}b^{-1/3}\gamma$ is much smaller than π/t_R, the fundamental frequency in the delay-induced instability.[5,6]

The simple periodic solution (5a), (5b) is, however, not always stable. Using a computer we have found that a solution with a single period undergoes successive bifurcations and finally gets into a chaotic state when parameters a and b are suitably varied. Figure 1 shows a typical example of hysteresis among stationary, periodic, and chaotic states obtained by varying a for fixed b. Figure 2 is a phase diagram of the states that appear when a is slowly decreased from a sufficiently large value.[9] If a is decreased beyond the bold

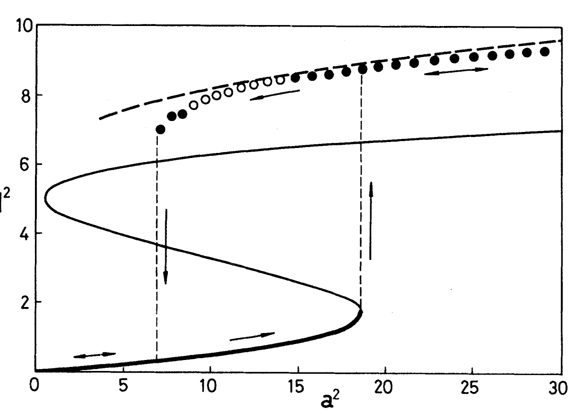

FIG. 1. $|\xi_s|^2$ vs a^2 relation (solid curve) and hysteresis among stationary, periodic, and chaotic states for $b=0.3$ and $\eta_0=5$. The solid and open circles indicate the long-time average of $|\xi|^2$ in periodic and chaotic states, respectively. The broken curve indicates the average calculated from Eq. (5a).

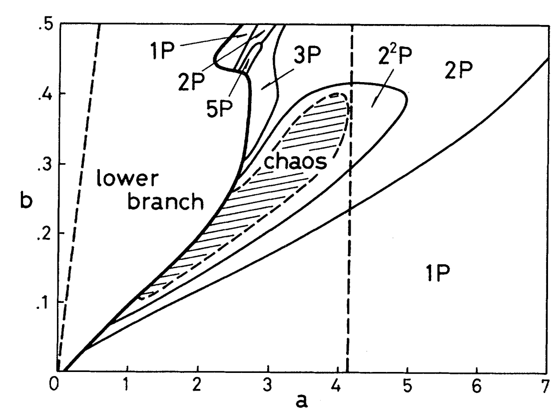

FIG. 2. Phase diagram of the states that appear when a is slowly decreased from a sufficiently large value ($\eta_0=5$). Symbol nP denotes the domain of limit cycles with period n. The broken lines indicate the upper and lower limits of the threefold range in the $|\xi_s|^2$ vs a^2 relation.

curve in Fig. 2, a jump occurs to the stationary state on the lower branch, no stationary state on the upper branch being realized. The domain of chaotic states is surrounded by the domains of limit cycles with period 2^n as usual.[10] The power of the output electric field from the cavity therefore exhibits period doubling (or successive subharmonic) bifurcations in the vicinity of the boundary of the chaotic domain, as seen in Figs. 3(a)–3(e). The time average of the output power and the fundamental frequency in the chaotic regime agree considerably well with those obtained by Eq. (5a), the values for the limit cycle with a single period (Fig. 1). Further, the chaotic states appear only for values of a for which the $|\xi_s|^2$ vs a^2 relation is threefold. These facts enable us to interpret the chaos in our system as a self-induced Rabi nutation subjected to a disturbance; the disturbance is considered to be caused by the unstable stationary state on the intermediate (negative-slope) branch.

Regarding Eq. (2a) as an optical Bloch equation, we see that Eqs. (2a) and (2b) have resemblance to the laser rate equation[11] or the Lorenz equation.[12] The Lorenz equation, however, admits no multiple stationary solution. Therefore, the chaos in our system should have an origin quite different from the Lorenz chaos.

Unlike the *strong* or *fully developed* turbulence seen in the delay-induced instability,[5] the chaotic state described by Eqs. (2a) and (2b) is to be called a *weak* turbulence (or chaos). In the turbulence of this type, which is peculiar to three-dimensional dissipative systems,[12,13] the orbit in phase space is restricted on a quasi two-dimensional invariant manifold [Fig. 4(a)]. This fact is in our case well understood by using a topologically equivalent model illustrated in Figs. 4(b)–4(d). The rotation along the Möbius sheet corresponds to the Rabi nutation around an unstable stationary state on the upper branch of the $|\xi_s|^2$ vs a^2 relation. The chaotic behavior of the orbit originates from an unlimited deformation of the sheet, i.e., an endless repetition of expansion [Fig. 4(b)] and folding [Fig. 4(c)] in the direction perpendicular to the rotation.

In conclusion, we have pointed out the possibility of a new kind of instability in an optical cavity having multiple stationary states. This instabil-

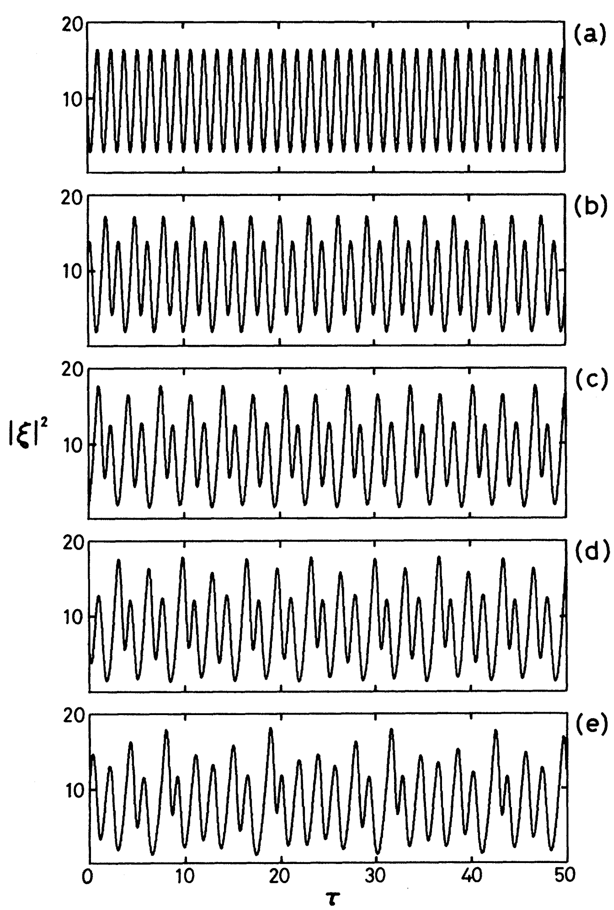

FIG. 3. Period-doubling bifurcations of self-pulsation for $b=0.3$ and $\eta_0=5$: (a) period 1 ($a=5.4$), (b) period 2 ($a=4.6$), (c) period 4 ($a=4.0$), (d) period 8 ($a=3.84$), and (e) chaotic ($a=3.5$).

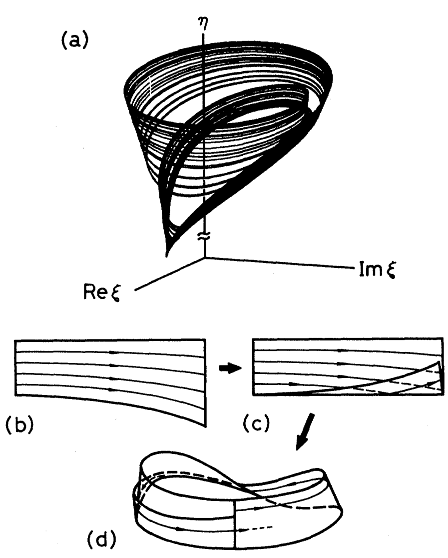

FIG. 4. (a) Orbit of a chaotic solution in phase space. (b)–(d) A topological illustration of the formation of a quasi two-dimensional manifold on which the chaotic orbit is restricted.

ity takes place when the relaxation time of the medium is shorter than the lifetime of the cavity, i.e., $1-R+\alpha l/2<t_R\gamma$,[7] and can be interpreted as a self-induced Rabi nutation of the electric field vector. For a Fabry-Perot cavity containing a Kerr medium, the incident power required for the occurrence of this instability is given by $[4a^2(t_R\gamma)^3/(1-R)kln_2]\times10^{-5}$ MW/cm^2; its typical value is estimated to be 10 MW/cm^2 by using $n_2\sim10^{-11}$ esu,[14] $l\sim1.5$ cm, $k\sim10^5$ cm^{-1}, $t_R\gamma\sim0.1$, and $R=0.97$. Thus the instability discussed in this Letter will be observed for a smaller incident power as compared with the delay-induced instability, if a medium of the same length is used.[5] The experiment will be free from transverse focusing effects, because the induced phase phase shift is sufficiently small.

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[9]This means the following: For a sufficiently large a, say a_1, Eqs. (2a) and (2b) are solved with increasing τ, starting from arbitrary $\xi(0)$ and $\eta(0)$. At a sufficiently large τ, say τ_1, a is decreased slightly. Starting from $\xi(\tau_1;a_1)$ and $\eta(\tau_1;a_1)$, the equations are solved again with increasing τ. The latter two steps are repeated. As the solutions in the vicinity of the chaotic domain in Fig. 2 depend sensitively on the initial condition, a different phase diagram will be obtained if the equations are solved in a different manner.

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Tandem Transport and Ambipolarity in the Resonant Plateau Regime

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A simple model for ion radial transport in the resonant plateau regime is derived. It is shown that the ion diffusion coefficient is insensitive to the radial electric field over a wide range of parameters. With use of this result, and the expression for the electron flux, a self-consistent picture of ambipolar tandem operation for quadrupole symmetric systems is obtained and compared with experiment.

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The success of the tandem mirror[1] as a viable fusion reactor depends in part on keeping the radial particle loss rate small compared to end losses, in order that the gain of electrostatic end plugging not be outweighed by radial transport. Additionally, the radial variation of the electrostatic potential, which, as will be seen, is coupled to the radial transport problem, is itself of central concern since the tandem concept requires a confining potential for solenoid ions on field lines away from, as well as along, the machine axis. In this Letter, a simple model for the radial ion flux in the resonant plateau (RP) regime is derived. With use of a previously obtained result for the electron flux,[2] a self-consistent picture of ambipolar tandem operation is obtained.

