# Fixed-Point Structure of $(\varphi^6)_3$ at Large $N$

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For a $N$-component isovector field $\varphi$, $(\varphi^6)_3$ is free in the infrared limit for all $N$. If $N$ is sufficiently large there is also an ultraviolet stable fixed point, calculable within perturbation theory, at which anomalous dimensions are $\sim N^{-1}$.

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The existence of logarithmic corrections to scaling about a tricritical point in three dimensions is well known.[1,2] Here I consider the corresponding field theory, $(\varphi^6)_3$, for its intrinsic interest. When $N$ is large, I demonstrate that $(\varphi^6)_3$ provides a *unique* example of a field theory, with no dimensional couplings,[3] which has a nontrivial ultraviolet limit calculable from perturbation theory by use of the renormalization group.

In three (Euclidean) dimensions, the Lagrangian density is

$$\mathcal{L}=Z_\varphi\frac{(\partial_i\overline{\varphi})^2}{2}+Z\,\frac{\pi^2}{3}\,\lambda(\overline{\varphi}^2)^3 + Z_{\varphi^2}\frac{\overline{\varphi}^2}{2}J_{\varphi^2}+Z_{\varphi^4}\frac{(\overline{\varphi}^2)^2}{4!}J_{\varphi^4}\,. \tag{1}$$

$\overline{\varphi}$ is a $N$-component isovector. The coefficient of the dimensionless coupling $\lambda$ is chosen in anticipation of the final results. The couplings for $\varphi^2$ and $\varphi^4$ are adjusted to vanish exactly. In the tricritical phase diagram, this corresponds to sitting right on the tricritical point. $J_{\varphi^2}$ and $J_{\varphi^4}$ are sources for single[4] insertions of the operators $\varphi^2$ and $\varphi^4$.

I renormalize the theory by using dimensional regularization in $3-\epsilon$ dimensions with minimal subtraction.[5] This renormalization scheme is particularly convienient since once the $\varphi^2$ and $\varphi^4$ couplings are set to zero at the tree level, radiative corrections will not alter them. In $3-\epsilon$ dimensions, the coupling $\lambda$ should be replaced by $\lambda\mu^{2\epsilon}$, where $\mu$ is a mass scale at which the values of the renormalized Green's functions are defined.

The calculations I perform essentially duplicate those of Lewis and Adams,[6] who determined tricritical exponents in $3-\epsilon$ dimensions at next to leading order in $\epsilon$. Nevertheless, because of the dependence of the $\beta$ function in $3-\epsilon$ dimensions on the renormalization scheme,[7] an independent calculation was necessary. Further, what is of general interest are the conclusions I draw concerning the $N\to\infty$ limit and the validity of the $\epsilon$ expansion, which are original.

By standard methods[8]

$$\beta=-\epsilon\left[\frac{\partial}{\partial\lambda}\ln\left(\frac{Z\lambda}{Z_\varphi{}^3}\right)\right]^{-1}=b_1\lambda^2-b_2\lambda^3\,,$$

$$b_1=3N+22\,,$$

$$b_2=\tfrac{1}{32}\pi^2(N^3+34N^2+620N+2720)+\tfrac{1}{4}(53N^2+858N+3304)\,. \tag{2a}$$

$$\gamma_\varphi=\beta\frac{\partial}{\partial\lambda}\ln Z_\varphi=\frac{(N+2)(N+4)}{48}\lambda^2[1-(N+\tfrac{22}{3})\lambda]\,. \tag{2b}$$

$$\gamma_{\varphi^2}=\beta\frac{\partial}{\partial\lambda}\ln Z_{\varphi^2}=-\frac{5(N+2)(N+4)}{16}\lambda^2(1-c_1\lambda)\,,$$

$$c_1=\tfrac{1}{8}\pi^2(N+4)(N+14)+\tfrac{11}{5}(3N+22)\,. \tag{2c}$$

$$\gamma_{\varphi^4}=\beta\frac{\partial}{\partial\lambda}\ln Z_{\varphi^4}=-2(N+4)\lambda(1-c_2\lambda)\,,$$

$$c_2=\tfrac{1}{64}\pi^2(N^2+18N+116)+\tfrac{3}{16}(19N+126)\,. \tag{2d}$$

Changes in the renormalization scheme alter $\lambda$ as $\lambda'=\lambda+O(\lambda^2)$. In the usual manner, it can be shown from their definitions that the first two terms of $\beta$, but only the first terms of the $\gamma$'s, are left invariant under $\lambda\to\lambda'$.

In the infrared limit the coupling vanishes logarithmically. Equation (2a) can be used to find the leading universal correction to this approach:

$$\lambda(p)\underset{p\ll\mu}{\sim}\frac{b_1{}^{-1}}{-\ln p/\mu}+\frac{b_2}{b_1{}^3}\frac{\ln(-\ln p/\mu)}{(\ln p/\mu)^2}+O((\ln p/\mu)^{-2})\,. \tag{3}$$

There is an ultraviolet stable fixed point $\lambda^*$ for all $N$. For small $N$, $\lambda^*$ could be an illusion of perturbation theory. In contrast, as $N\to\infty$,

$$\lambda^*=\frac{96}{\pi^2}\frac{1}{N^2},\quad \gamma_\varphi{}^*=\frac{192}{\pi^4}\frac{1}{N^2},$$
$$\gamma_{\varphi^2}{}^*=\frac{31680}{\pi^4}\frac{1}{N^2},\quad \gamma_{\varphi^4}{}^*=\frac{96}{\pi^2}\frac{1}{N}, \tag{4a}$$

where $\gamma^*=\gamma(\lambda^*)$, and

$$\lambda(p)\underset{p\gg\mu}{\sim}\lambda^*-\frac{\delta\lambda}{(p/\mu)^{288/\pi^2N}}, \tag{4b}$$

$\delta\lambda=\lambda^*-\lambda(\mu)$, $\delta\lambda>0$. When $N\to\infty$, $\lambda^*$ and the $\gamma^*$ vanish, and so assuredly $\lambda^*$ exists for large $N$.[9] Since the asymptotic values of Eq. (4a) agree with the results obtained from Eq. (2) to $\lesssim 16\%$ when $N\gtrsim 1000$, I presume that $\lambda^*$ exists at least when $N\gtrsim 1000$. Within perturbation theory, a better estimate could be given by calculating the corrections in $N^{-1}$ to Eq. (4).

To illustrate the nature of the theory for large $N$, consider the graphs contributing to Eq. (4). $\lambda^*$ is $\sim N^{-2}$ as $N\to\infty$ because there is a single graph, shown in Fig. 1, which contributes at $\sim N^3\lambda^3$ to $\beta$. $\gamma_{\varphi^4}{}^*$ is a sum of two terms, $\sim N\lambda$ and $\sim N^3\lambda^2$, where the term $\sim N^3\lambda^2$ is obtained from a single graph related to Fig. 1. $\gamma_{\varphi^2}{}^*$ is also a sum of two terms, $\sim N^2\lambda^2$ and $\sim N^4\lambda^3$, with four graphs contributing at $\sim N^4\lambda^3$. While two of the graphs $\sim N^4\lambda^3$ are related to Fig. 1, two are topologically distinct. One is shown in Fig. 2; the other is a finite graph $\sim N^3\lambda^2$ with a $\lambda$ counterterm. $\gamma_\varphi{}^*$ is rather uninteresting, as the leading term in $\lambda$, $\sim N^2\lambda^2$, is the only term $\sim N^{-2}$ as $N\to\infty$. As could be expected physically, all contributions to Eq. (4) are independent of the renormalization scheme.

Including terms of higher order in $\lambda$ in Eq. (2) affects Eq. (4) at higher order in $N^{-1}$. This would not be true if as $N\to\infty$ there were graphs with an additional factor of $\sim N^2$ for each new order in $\lambda$. Fortunately, graphs only increase as $\sim N^2$ at every other order in $\lambda$, and otherwise $\sim N$. For example, while for $\beta$ there are graphs $\sim N\lambda^2$ and $\sim N^3\lambda^3$, it can be shown that the graphs at higher order are at best $\sim N^4\lambda^4$ and $\sim N^6\lambda^5$. These terms at $\sim\lambda^4$ and $\sim\lambda^5$ simply generate corrections $\sim N^{-3}$ to the $\lambda^*$ of Eq. (4). Similarly, terms $\sim\lambda^3$ in $\gamma_{\varphi^4}$ and $\sim\lambda^4$ in $\gamma_\varphi$ and $\gamma_{\varphi^2}$ contribute $\sim N^{-2}$ to $\gamma_{\varphi^4}{}^*$ and $\sim N^{-3}$ to $\gamma_\varphi{}^*$ and $\gamma_{\varphi^2}{}^*$ as $N\to\infty$.

Restricting $\lambda$ to be $\leqslant\lambda^*\sim N^{-2}$, the infrared limit of Eq. (3) should be $p\ll\mu\exp(-N^{1+\alpha})$, $\alpha>0$. Consequently, logarithmic effects vanish in the infrared limit as $N\to\infty$, contrary to the conclusion at leading order in $\lambda$.[2] This is in agreement with general arguments by Emery.[10] Having demonstrated $\lambda\sim N^{-2}$, a $N^{-1}$ expansion can then be used to determine the tricritical phase diagram about $N=\infty$, or equivalently, the effective potential.[11]

In $3-\epsilon$ dimensions, with minimal subtraction,

$$\beta_{3-\epsilon}(\lambda)=-2\epsilon\lambda+b_1\lambda^2-b_2\lambda^3, \tag{5}$$

$\gamma_{3-\epsilon}(\lambda)=\gamma(\lambda)$. There are three fixed points when $\epsilon\ll 1$: an infrared stable point $\lambda_{ir}{}^*(\epsilon)\sim 2\epsilon/b_1+O(\epsilon^2)$, and ultraviolet stable points at $\lambda=0$ and $\lambda_{uv}{}^*(\epsilon)=\lambda^*-\lambda_{ir}{}^*(\epsilon)+O(\epsilon^2)$. As $\epsilon$ increases, so does $\lambda_{ir}{}^*(\epsilon)$ while $\lambda_{uv}{}^*(\epsilon)$ decreases. For all $N$ there is an $\epsilon_c$ at which $\lambda_{ir}{}^*(\epsilon_c)=\lambda_{uv}{}^*(\epsilon_c)$, where $\lambda^*(\epsilon_c)$ has marginal stability. When $\epsilon>\epsilon_c$, $\lambda_{ir}{}^*(\epsilon)$ and $\lambda_{uv}{}^*(\epsilon)$ are complex, and the only fixed point is $\lambda=0$. As $N\to\infty$,

$$\epsilon_c=\frac{36}{\pi^2}\frac{1}{N}, \tag{6}$$

independent of the renormalization scheme. In this way, the existence of $\lambda^*$, and thereby $\epsilon_c$, provides a natural limit on the radius of convergence for the $\epsilon$ expansion at $\lambda_{ir}{}^*(\epsilon)$. What controls tricritical behavior when $\epsilon>\epsilon_c$, such as at $\epsilon=1$, is unclear.

It is reasonable to speculate that $\lambda^*$ and $\epsilon_c$ exist for all $N\geqslant 0$, not just $N\gtrsim 1000$. In support of this, it is found that if Eq. (2) is used blindly, the $\gamma^*$ are small for every $N\geqslant 0$. The largest $|\gamma^*|$ is usually $|\gamma_{\varphi^4}{}^*|$, whose maximum is only $\gamma_{\varphi^4}{}^*\sim -0.053$ at $N=3$. In addition, the largest $\epsilon_c$ is merely $\epsilon_c\sim 0.040$ at $N=10$. Thus $\epsilon_c$ is very possibly $<1$ whenever $N\geqslant 0$. Of course, for small $N\lambda^*$, the $\gamma^*$ and $\epsilon_c$ do depend on the renormalization scheme.

What is needed to study $\lambda^*$ and $\epsilon_c$ for small $N$ are strong-coupling techniques. Approximate recursion relations have been used by Wilson[12]

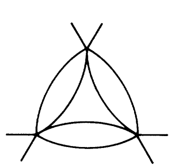

FIG. 1. The only graph $\sim N^3\lambda^3$ in $\beta$.

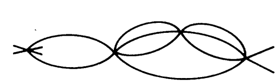

FIG. 2. A graph $\sim N^4\lambda^3$ in $\gamma_{\varphi^2}$. The cross denotes an insertion of the operator $\varphi^2$.

for $(\varphi^6)_3$ at $N=1$. He finds a renormalized theory with interaction, but one very different from perturbation theory: although $\gamma_{\varphi^2} \neq 0$, there is no wave-function or coupling-constant renormalization! Clearly further study would be helpful.

Lastly, it is hoped that $(\varphi^6)_3$ can serve as a test for methods used to determine the existence of a renormalized $(\varphi^4)_4$ theory. Evidence for a $\lambda^*$ in $(\varphi^4)_4$ has been presented,[13] but is not seen by approximate recursion relations,[12,14] strong-coupling expansions,[14,15] or Monte Carlo simulations.[16] $(\varphi^6)_3$ is admittedly very different from $(\varphi^4)_4$,[17] but its example demonstrates that there is *a priori* no logical necessity for a renormalized $(\varphi^4)_4$ theory to be free.

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[3]Any model which has a dimensionless coupling and is asymptotically free in $D$ dimensions will have an ultraviolet stable fixed point $\sim O(\epsilon)$ in $D+\epsilon$ dimensions. This includes the nonlinear $\sigma$ model in $2+\epsilon$ dimensions and Yang-Mills fields in $4+\epsilon$ dimensions. Contrary to $(\varphi^6)_3$, when $\epsilon > 0$ the coupling in these models acquires dimension. A model whose behavior is converse to that of $(\varphi^6)_3$ at large $N$—asymptotic freedom with a calculable infrared stable fixed point—is four-dimensional Yang-Mills fields with sufficiently many fermions [W. Caswell, Phys. Rev. Lett. 33, 244 (1974)].

[4]Another counterterm proportional to $\overline{\varphi}^2$ is needed to make multiple insertions of $\varphi^4$ finite [see, e.g., *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. Green (Academic, New York, 1976), Sec. VIIID].

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[6]A. L. Lewis and F. W. Adams, Phys. Rev. B 18, 5099 (1978). To leading order in $\epsilon$, see M. J. Stephen and J. L. McCauley, Jr., Phys. Lett. A 44, 89 (1973). In $3-\epsilon$ dimensions, the critical exponents $\eta$, $\gamma$, and $\varphi$ are related to the $\gamma$'s at $\lambda_{ir}^*(\epsilon)$ [Eq. (5)] by $\eta=\gamma_\varphi^*$, $\gamma=(2-\eta)/(2-\eta+\gamma_{\varphi^2}^*)$, $\varphi=\gamma(1+\epsilon-2\eta)/(2-\eta)+\frac{1}{2}\gamma\gamma_{\varphi^4}^*$. The new result of Eq. (2) is $\gamma_{\varphi^2}$ to $\sim O(\lambda^3)$, which yields $\gamma$ to $\sim O(\epsilon^3)$.

[7]In $3-\epsilon$ dimensions, $\sim O(\lambda^3)$ is the third term in $\beta$ [Eq. (5)], and thus can depend on the renormalization scheme. $\lambda_{ir}^*(\epsilon)$ to $\sim O(\epsilon^2)$ with minimal subtraction is obtained from Eq. (8) of Lewis and Adams (Ref. 6) by setting their cutoff-dependent constant $C$ equal to zero.

[8]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. Green (Academic, New York, 1976).

[9]With use of a $N^{-1}$ expansion about $N=\infty$, the existence of $\lambda^*$ in $(\varphi^6)_3$ has been noted by P. K. Townsend, Nucl. Phys. B 118, 199 (1977); and T. Appelquist and U. Heinz, Yale University Report No. YTP82-01 (to be published). These authors assert that $\lambda^*$ is inaccessible from perturbation theory in $\lambda$, and can only be seen by a $N^{-1}$ expansion. In contrast, when $N$ is large I determine $\lambda^*$ directly from perturbation theory. The $\lambda^*$'s found by these authors do not agree with the value of Eq. (4a); they do not evaluate any $\gamma^*$. I would like to thank T. Appelquist for sending me a draft of the above, which was received after this manuscript was completed.

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[17]For example, with use of $\beta$ for $(\varphi^4)_4$ to the same order as Eq. (5) (Ref. 8) $\epsilon_c \sim 0.381$ at $N=0$, increasing monotonically to $\sim N/36$ as $N\to\infty$. This is reassuring, since surely $\epsilon_c$, if it exists in $(\varphi^4)_{4-\epsilon}$, is $>1$ for all $N \geq 0$.