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Exact First-Order Electron Self-Energy Contribution to the Decay Rate of Orthopositronium

Michael A. Stroscio
Air Force Office of Scientific Research, Bolling Air Force Base, Washington, D. C. 20332
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There is presently a discrepancy between the theoretical and the experimental determinations of the purely quantum electrodynamic decay rate for orthopositronium. This Letter presents the first exact calculation of any contribution to the decay rate of orthopositronium to order α^7.

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The decay rate of orthopositronium is of special interest in quantum electrodynamics since it is defined to high accuracy by purely electrodynamic interactions. In this paper the exact electron self-energy contribution to this decay rate is formulated and explicitly derived in the Feynman gauge.

The decay rate of orthopositronium is of the form[1]

$$\Gamma=\frac{1}{72\pi}\frac{\alpha^6 mc^2}{\hbar}\int_0^1 dK_1\int_{1-K_1}^1 dK_2 X(K_1,K_2), \qquad (1)$$

where $X(K_1,K_2)$ represents the decay matrix, $K_i=\hbar\omega_i/mc^2$, α is the fine-structure constant, $\hbar$ is Planck's constant, $\hbar\omega_i$ is the energy of photon i, and mc^2 is the electron rest energy. The lowest-order, nonradiative decay rate is[1]

$$\Gamma_a=\frac{2}{9\pi}\frac{\alpha^6 mc^2}{\hbar}(\pi^2-9). \qquad (2)$$

The lowest-order radiative corrections to this result were first calculated numerically[2,3] by Stroscio and Holt and later by Caswell and co-workers,[4,5] who found significant corrections to the binding-energy term.[6]

The Feynman diagrams for the nonradiative, α^6, and electron self-energy, α^7, contributions to the orthopositronium decay rate are shown in Fig. 1. The exact matrix element and the decay rate for the electron self-energy contribution were formulated in the Feynman gauge and are

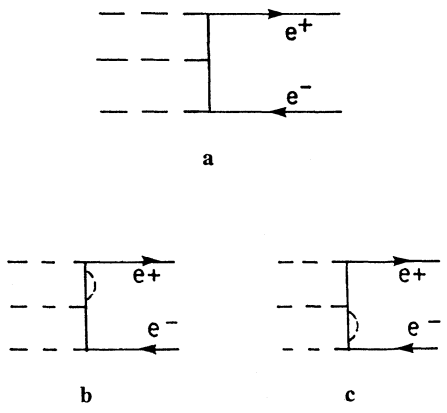

FIG. 1. Feynman diagrams for the nonradiative (α^6) and the electron self-energy (α^7) contributions to the decay rate of orthopositronium. The initial state contains an electron, e^-, and a positron, e^+, while the final state contains three photons of frequencies ω_1, ω_2, and ω_3.

given by[2]

$$\Gamma_{bc}=\frac{1}{72\pi}\frac{\alpha^6mc^2}{\hbar}\int_0^1 dK_1\int_{1-K_1}^1 dK_2\, X_{bc}(K_1,K_2), \tag{3}$$

where

$$X_{bc}=\frac{2}{(K_1K_2K_3)^2}\sum^{6\,\text{perm.}}[P_1(K_2,K_3)\Sigma_1(K_1)+P_2(K_2,K_3)\Sigma_2(K_1)], \tag{4}$$

with $\Sigma_1(K_1)$ and $\Sigma_2(K_1)$ being the standard electron self-energy parts[2,7] and[2]

$$P_1(K_2,K_3)=-16K_2+24K_2^2+36K_2K_3-8K_2^3-56K_2^2K_3+16K_2^2K_3^2+4K_2^3K_3, \tag{5}$$

$$P_2(K_2,K_3)=16K_2^2+16K_2K_3-32K_2^3-56K_2^2K_3+16K_2^4+36K_2^2K_3^2+28K_2^3K_3-4K_2^4K_3-20K_2^3K_3^2. \tag{6}$$

In Eq. (4) the summation over the six permutations corresponds to the six permutations of ω_1, ω_2, and ω_3 in the final state of the diagrams of Fig. 1. Herein, λ/m represents the photon mass divided by the electron mass and is included to isolate the infrared-divergent terms in Γ_{bc}.

The integrals in Γ_{bc} are of the form

$$I=\int_0^1 dK_1\int_{1-K_1}^1 dK_2 X_{bc}(K_1,K_2,K_3=2-K_1-K_2). \tag{7}$$

Using the energy-conservation condition, $K_3=2-K_1-K_2$, and changing variables it follows that K_1, K_2, and K_3 may be interchanged under the integral in Eq. (7) without changing the value of I. This result is a consequence of the limits of integration on K_1 and K_2 and allows the sum in Eq. (4) to be omitted upon multiplying Eq. (4) by a factor of 6. Then

$$\Gamma_{bc}=\frac{\alpha^6mc^2}{12\pi\hbar}\frac{\alpha}{\pi}\{-16I_1^{10}+24I_1^{20}+36I_1^{11}-8I_1^{30}-56I_1^{21}+16I_1^{22}+4I_1^{31}+16I_2^{20}+16I_2^{11}-32I_2^{30}-56I_2^{21}$$
$$+16I_2^{40}+36I_2^{22}+28I_2^{31}-4I_2^{41}-20I_2^{32}\}+\frac{8}{9}\frac{\alpha^6mc^2}{\pi\hbar}\frac{\alpha}{\pi}\left[(\pi^2-9)\ln\left(\frac{\lambda}{m}\right)\right]. \tag{8}$$

The integrals I_1^{nm} and I_2^{nm} are

$$I_1^{nm}=\int_0^1 dK_1\int_{1-K_1}^1 dK_2\frac{K_1K_2^nK_3^m}{(K_1K_2K_3)^2}\frac{1}{1-2K_1}\left(1-\frac{2-6K_1}{1-2K_1}\ln(2K_1)\right), \tag{9}$$

$$I_2^{nm}=\int_0^1 dK_1\int_{1-K_1}^1 dK_2\frac{K_2^nK_3^m}{(K_1K_2K_3)^2}\frac{1}{1-2K_1}\left(-2+3K_1+\frac{2-4K_1-2K_1^2}{1-2K_1}\ln(2K_1)\right). \tag{10}$$

These integrals have been evaluated and the results are given in Table I.

With Eq. (8) and the results of Table I,

$$\Gamma_{bc}=\frac{\alpha^7mc^2}{\pi^2\hbar}\left\{-\frac{7}{6}+\frac{805}{1296}\pi^2+\frac{67}{18}\ln 2-\frac{775}{1944}\pi^2\ln 2-\frac{1049}{324}\zeta(3)\right\}+\frac{\alpha^7mc^2}{\pi^2\hbar}\left[\frac{8}{9}(\pi^2-9)\ln\left(\frac{\lambda}{m}\right)\right], \tag{11}$$

where $\zeta(3)$ is the Riemann-zeta function of argument 3 that results from the Spence functions[8] frequently encountered in quantum electrodynamic calculations.

Previous numerical approximations[2,4] to Eq. (11) have been expressed in terms of the ratio Γ_{bc}/Γ_a. From Eqs. (2) and (11),

$$\frac{\Gamma_{bc}}{\Gamma_a}=\left[4.784\,98+4\ln\left(\frac{\lambda}{m}\right)\right]\left(\frac{\alpha}{\pi}\right). \tag{12}$$

This numerical coefficient agrees with the value 4.785 ± 0.010 of Stroscio and Holt.[2] In fact, a numerical integration of Eq. (8) by Gauss-Legendre techniques yields, for the numerical coefficient of Eq. (12), the values 4.784 85 and 4.784 93 for 24- and 32-point integrations, respectively. The 32-point result agrees with the exact result of Eq. (11) to approximately 1 part in 10^5. This is excellent agreement. On the other hand, the numerical integration of Caswell, Lepage, and Sapirstein[4] resulted in the numerical coefficient 4.791 ± 0.003. Both the exact and the precise numerical values of the present work disagree with the result of Ref. 4.

In summary, the exact electron self-energy contribution to the decay rate of orthopositronium to order α^7 has been derived in the Feynman

TABLE I. The integrals I_1^{nm} and I_2^{nm} are defined by multiplying the numerical coefficients in the column under a particular integral name by the factors 1, π^2, ln2, π^2ln2, $\zeta(3)$, lnλ, ln2 lnλ, and ln$^2\lambda$. For example, $I_1^{21} = \frac{1}{12}\pi^2 + \frac{1}{6}\pi^2\ln 2 + 2\zeta(3)$. λ replaces λ/m for brevity.

	I_1^{10}	I_1^{20}	I_1^{11}	I_1^{30}	I_1^{21}	I_1^{22}	I_1^{31}	I_2^{20}	I_2^{11}	I_2^{30}	I_2^{21}	I_2^{40}	I_2^{22}	I_2^{31}	I_2^{41}	I_2^{32}
1												1			-2	1/2
π^2	$\frac{17}{216}$	$\frac{1}{3}$	$\frac{1}{36}$	$\frac{5}{8}$	$\frac{1}{12}$	$\frac{3}{8}$	$\frac{-1}{6}$	$\frac{-5}{6}$	$\frac{-11}{18}$	$\frac{-25}{24}$	$\frac{-5}{12}$	$\frac{-45}{24}$	$\frac{-5}{8}$	$\frac{1}{12}$	$\frac{3}{4}$	$\frac{-7}{16}$
ln2	$\frac{-4}{3}$	-2		-3		-1	1	2	2	1	2	-2	1	3	$\frac{9}{2}$	1
π^2ln2	$\frac{143}{162}$		$\frac{22}{27}$	$\frac{-1}{6}$	$\frac{1}{6}$		$\frac{-1}{24}$		$\frac{-47}{54}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{3}{4}$		$\frac{-3}{8}$	$\frac{-13}{24}$	
$\zeta(3)$	$\frac{-265}{216}$		$\frac{7}{36}$	-2	2		4		$\frac{-269}{72}$	4	-4	12		-6	-8	
lnλ	1	1		1				3	2	3	2	3	2	2	2	2
ln2lnλ	-4	-4		-4				2	-2	2	-2	2	-2	-2	-2	-2
ln$^2\lambda$								-1	-1	-1	-1	-1	-1	-1	-1	-1

gauge. This result provides the first rigorous test of the theoretical uncertainty estimates of previous numerical calculations.[2,4] The precise estimation of numerical uncertainty in many numerical integration routines is very difficult to accomplish. In addition, in deriving the exact electron self-energy corrections, symmetry conditions have been identified. This result provides a benchmark for checking the correctness of symbol-manipulating algorithms that are designed to evaluate integrals analytically. This is especially so since the basic integrals in all of the α^7 contributions to the orthopositronium decay rate are of the same origin and same relationship to the Spence functions. Finally, this result provides the first exact calculation of any α^7 contribution to the decay rate of orthopositronium and there is presently a discrepancy between the theoretical and the experimental[9] determinations for this purely quantum electrodynamic process.

[1]A. Ore and J. L. Powell, Phys. Rev. 75, 1696 (1949).
[2]Michael A. Stroscio and John M. Holt, Phys. Rev. A 10, 749 (1974).
[3]Michael A. Stroscio, Phys. Rep. 22C, 215 (1975).
[4]W. E. Caswell, G. P. Lepage, and J. Sapirstein, Phys. Rev. Lett. 38, 488 (1977).
[5]W. E. Caswell and G. P. Lepage, Phys. Rev. A 20, 36 (1979).
[6]A. Rich, Rev. Mod. Phys. 53, 127 (1981).
[7]J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Mass., 1955).
[8]K. S. Kolbig, J. A. Mignaco, and E. Remiddi, CERN Report No. CERN-DD-CO-69-5, 1969 (unpublished); H. F. Sandham, Proc. London Math. Soc. 83 (1949).
[9]Ref. 6 provides an excellent review of the current status with regard to experimental determinations of the decay rate of orthopositronium.