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New Interpretation of the Scalar Product in Hilbert Space

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A new interpretation of the scalar product in Hilbert space was presented recently by Aharonov, Albert, and Au. Here the essential equivalence between such a viewpoint and Wigner's exact reformulation of nonrelativistic quantum mechanics using distribution functions is pointed out.

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Aharonov, Albert, and Au[1] have recently presented a new interpretation of the scalar product of two states in Hilbert space in terms of a relationship between two or more distinct physical systems. In particular, whereas the conventional viewpoint is that the scalar product gives the projection of one state on another in *Hilbert space*, in the work of Aharonov, Albert, and Au it appears as a measure of the proximity of the two states in *phase space*.

Our purpose here is to point out that such a viewpoint is closely related to the work of Wigner,[2] who introduced his now-famous quantum mechanical distribution function by means of which one obtains an exact reformulation of nonrelativistic quantum mechanics.

The distribution functions $P(q,p)$ are functions of position and momentum coordinates q and p. In the classical limit, $P(q,p)$ is the phase-space distribution function which gives the probability that the coordinates and momenta have the values q and p. In general, $P(q,p)$ depends on $\hbar$ and may assume negative values,[2,3] which accounts for the frequent description of this quantity as a quasiclassical distribution function.

In one dimension the distribution function may be written in the form

$$P(q,p) = (\pi\hbar)^{-1}\int \langle q-y|\rho|q+y\rangle e^{2ipy/\hbar}dy = (\pi\hbar)^{-1}\int \psi^*(q+y)\psi(q-y)e^{2ipy/\hbar}dy, \quad (1)$$

where ρ is the density matrix. As first shown in Ref. 3, and discussed at length by O'Connell and Wigner,[4,5] this function (as distinct from other distribution functions which have been proposed[5]) has the property that the transition probability between two states ψ and φ is given, in terms of the corresponding distribution functions, P_ψ and P_φ say, as follows:

$$(2\pi\hbar)^{-1}\left|\int \psi(x)^*\varphi(x)dx\right|^2 = \int\int P_\psi(q,p)P_\varphi(q,p)dq\,dp. \quad (2)$$

In other words, the scalar product in Hilbert space has been replaced by an integration in phase space. Further, since $P_\psi(p,q)$ can be interpreted as the probability that a particle in the state ψ has momentum p and coordinate q (even though such a probability cannot be directly measured), it follows that $P_\psi(p,q)P_\varphi(p,q)$ is the probability that two particles, one in the state ψ and the other in the state φ, have the same specific momenta p and the same specific coordinate

q. Hence, carrying out the phase-space integration over all p and q, we reach the conclusion that the left-hand side of (2), i.e., $(2\pi\hbar)^{-1}|\langle\psi|\varphi\rangle|^2$ is, in the words of Aharonov, Albert, and Au,[1] "... a propensity of two particles, one in the state $|\varphi\rangle$ and the other in the state $|\psi\rangle$, to have the same positions and the same momenta."

We turn now to a generalization of Eq. (2) by considering

$$J \equiv \int\int P_\psi(q,p)P_\varphi(q+\beta, p+\alpha)dp\,dq\,. \tag{3}$$

Using Eq. (1) in Eq. (3) and carrying out the p and one of the y integrations, we obtain

$$J = (\pi\hbar)^{-1}\int\int dy\,dq\,\psi^*(q+y)\psi(q-y)\varphi^*(q+\beta-y)\varphi(q+\beta+y)e^{2i\alpha y/\hbar}\,. \tag{4}$$

Changing variables by setting $q+y=u$ and $q-y=v$, we obtain

$$J = (2\pi\hbar)^{-1}\int\int du\,dv\,\psi^*(u)\psi(v)\varphi^*(v+\beta)\varphi(u+\beta)e^{i\alpha(u-v)/\hbar} = (2\pi\hbar)^{-1}\left|\int\psi^*(x)e^{i\alpha x/\hbar}\varphi(x+\beta)dx\right|^2. \tag{5}$$

Now using the fact that

$$e^{i\alpha x/\hbar}\varphi(x+\beta) = e^{i\alpha(\hat{x}-\beta)/\hbar}\varphi(x+\beta)\,, \tag{6}$$

and

$$\varphi(x+\beta) = e^{-i\beta\hat{p}/\hbar}\varphi(x)\,, \tag{7}$$

where $\hat{x}$ and $\hat{p}$ are operators, we obtain

$$J = (2\pi\hbar)^{-1}|\psi^*(x)e^{-i\alpha\beta/\hbar}e^{i\alpha\hat{x}/\hbar}e^{-i\beta\hat{p}/\hbar}\varphi(x)|^2$$
$$= (2\pi\hbar)^{-1}|\psi^*(x)e^{i(\alpha\hat{x}-\beta\hat{p})/\hbar}\varphi(x)|^2. \tag{8}$$

Thus, we conclude that

$$\int\int P_\psi(q,p)P_\varphi(q+\beta,p+\alpha)dp\,dq = (2\pi\hbar)^{-1}|\langle\psi|\alpha,\beta\rangle_\varphi|^2, \tag{9}$$

where, in the notation of Aharonov, Albert, and Au,[1]

$$|\alpha,\beta\rangle_\varphi = e^{i(\alpha\hat{x}-\beta\hat{p})/\hbar}|\varphi\rangle, \tag{10}$$

i.e., $|\varphi\rangle$ is translated through a distance β in coordinate space and α in momentum space. Hence we can conclude that $(2\pi h)^{-1}|\langle\psi|\alpha,\beta\rangle_\varphi|^2$ is a propensity of two particles, one in the state $|\varphi\rangle$ and the other in the state $|\psi\rangle$ to have positions and momenta differing by amounts β and α, respectively.[6] This corresponds to a result obtained by Aharonov, Albert, and Au[1] [see their Eq. (4)].

In summary, we have demonstrated that the new interpretation given by Aharonov, Albert, and Au[1] to the scalar product of two states in Hilbert space is essentially equivalent to the exact reformulation of nonrelativistic quantum mechanics in terms of distribution functions, which was pioneered by Wigner nearly one-half century ago.[2]

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