processes share the advantages of other optical techniques for surface study. Yet unlike optical methods, these processes can exhibit the high degree of intrinsic surface sensitivity characteristic of the techniques relying on the absorption, emission, or scattering of electrons and other massive particles.

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Stability of Electron Vortex Structures in Phase Space

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The stability of one-dimensional, solitary vortex structures in the electron phase space (electron holes) is investigated. A linear eigenvalue problem is derived in the fluid limit and solved exactly, assuming that the normal mode is well represented by the lowest eigenstate of a properly chosen field operator. A new dispersion relation is obtained which exhibits purely growing solutions in two dimensions but only marginally stable solutions in one dimension. This explains the numerically well-known fact that vortex structures disappear in going from one to two dimensions.

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Phase-space vortices, best known from particle simulations,[1] are saturated trapped-particle states of beam- or wave-driven plasmas. Only recently they have been observed experimentally[2] and described analytically.[3] The analytic solution in terms of a specially tailored Bernstein-Greene-Kruskal (BGK) wave shows a characteristic positive potential hump in which a finite number of electrons is trapped. For its existence there has to be a deficit of deeply trapped particles in accordance with the ring-shaped pattern of a vortex structure in phase space. Its characteristic speed is of the order of the electron thermal velocity or less. Being thus an entirely kinetic phenomenon, little information is available about its stability behavior. Stability theories[4] presented up to now deal with idealized periodic BGK equilibria.

This paper is devoted to the stability of a solitary, localized one whose physical existence has been proven experimentally. Guided by the general framework of Lewis and Symon,[5] I first derive an eigenvalue problem for two-dimensional perturbations adapted to localized BGK waves and then solve it under two assumptions.

An equilibrium phase-space vortex is described[3]

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in the weak-amplitude limit by an electric potential given by $\varphi_0(x)=\psi_0\,\mathrm{sech}^4(\alpha x)$, where ψ_0 is the normalized amplitude and $\alpha\sim\psi_0^{1/4}$, the proportionality constant depending on a parameter which reflects the state of trapped electrons. In the wave frame, the corresponding electron distribution function[3] $f_0(x,v_x)$ is a generalization of a shifted Maxwellian for untrapped particles and a Maxwellian with a negative "temperature" for trapped particles. The generalization consists of replacing the particle velocity v_x by $\sigma(2E_\parallel)^{1/2}$, where $\sigma=\mathrm{sgn}v_x$, and $E_\parallel=v_x^2/2-\varphi_0(x)$ is the normalized single-particle energy in the x direction. In the laboratory frame the speed is given by $v_0=1.3(1-16\alpha^2)$ which is normalized by the electron thermal speed. The electric potential energy, the coordinates, and the time are normalized by the electron thermal energy, the Debye length, and the inverse plasma frequency, respectively.

Two-dimensional perturbations to this equilibrium are governed by the Vlasov-Poisson system, assuming a fixed ion neutralizing background. After Fourier transformation in t and y the linearized system becomes

$$(-i\hat{\omega}+L)f_1=-(\partial_{E_\parallel}F_0L+ikv_y\,\partial_{E_\perp}F_0)\varphi_1,\qquad (\partial_x^2-k^2)\varphi_1=\int d^2v\,f_1, \tag{1}$$

where f_1 and φ_1 are the Fourier transforms of the perturbed electron distribution and the electric potential, respectively, k is the perpendicular wave number, and the background distribution reads in two dimensions $F_0=f_0(x,v_x)(2\pi)^{-1/2}\times\exp(-E_\perp)$, with $E_\perp=v_y^2/2$ the unperturbed particle energy in the perpendicular direction. $L=v_x\,\partial_x+\varphi_0'(x)\partial_{v_x}$ is the equilibrium Liouville operator, and $\hat{\omega}=\omega-kv_y$ is the Doppler-shifted frequency. By introducing both a new field operator[5]

$$\Lambda=\partial_x^2-k^2+\int d^2v\,\partial_{E_\parallel}F_0=\partial_x^2-k^2-n_e'(\varphi_0), \tag{2}$$

where n_e is the unperturbed electron density, and a new perturbed distribution function

$$g=f_1+\varphi_1\partial_{E_\parallel}F_0, \tag{3}$$

the system (1) transforms to

$$(-i\hat{\omega}+L)g=-i(\hat{\omega}\,\partial_{E_\parallel}+kv_y\,\partial_{E_\perp})F_0\varphi_1, \tag{4}$$

$$\Lambda\varphi_1=\int d^2v\,g. \tag{5}$$

Equation (4) is solved by the method of characteristics to give

$$g=-i(\hat{\omega}\partial_{E_\parallel}+kv_y\,\partial_{E_\perp})F_0\exp[i\hat{\omega}\tau_E(x,\sigma)]\int^x dx'\,\frac{\exp[-i\hat{\omega}\tau_E(x',\sigma)]}{v(x',E_\parallel,\sigma)}\,\varphi_1(x',k,\omega), \tag{6}$$

where $v(x,E_\parallel,\sigma)=\sigma\{2[E_\parallel+\varphi_0(x)]\}^{1/2}$ is the parallel particle velocity expressed in terms of $E_\parallel$ and σ.[4,5] In (6) $\tau_E(x,\sigma)\equiv\int_0^x dx'/v(x',E_\parallel,\sigma)$ is the time a particle takes to move from zero to x on an unperturbed orbit characterized by the constants of motion. Assuming that the perturbation is switched adiabatically, we get after repeated partial integrations

$$g=[\partial_{E_\parallel}+(kv_y/\hat{\omega})\partial_{E_\perp}]F_0\{1-(i/\hat{\omega})v_x\,\partial_x[1-(i/\hat{\omega})v_x\,\partial_x(1-\cdots)]\}\,\varphi_1(x,k,\omega). \tag{7}$$

Later on, in the so-called fluid limit, I cut this geometrical series assuming that the series is well represented by the leading terms.

φ_1, on the other hand, is expanded with respect to the eigenfunctions η of the field operator Λ which are given by

$$\Lambda(x,k)\eta_n=-\lambda_n\eta_n. \tag{8}$$

For the solitary vortex structures (8) turns out to be a solvable Schrödinger problem with a sech^2 potential. Λ possesses five discrete eigenstates besides the continuous spectrum, the lowest-energy state being given by

$$\eta_0(x)=\mathrm{sech}^5\alpha x,\quad \lambda_0=-9\alpha^2+k^2. \tag{9}$$

Assuming now that the normal mode is well described by (9), i.e., $\varphi_1(x,k,\omega)\sim\eta_0(x)$, we get after insertion of (7) into (5), multiplication of (5) by $\eta_0(x)$, and integration over x an eigenvalue equation of the following kind:

$$\lambda_0c_0+\int_{-\infty}^{+\infty}dx\,\eta_0(x)\int d^2v\left(\partial_{E_\parallel}+\frac{kv_y}{\hat{\omega}}\,\partial_{E_\perp}\right)F_0\left[1-\frac{iv_x\partial_x}{\hat{\omega}}-\frac{1}{\hat{\omega}^2}v_x^2\partial_x^2\right]\eta_0(x)=0, \tag{10}$$

where $c_0 = \int_{-\infty}^{+\infty} dx\, \mathrm{sech}^{10}(\alpha x) = 256/315\alpha$. In (10) I have assumed that fourth- and higher-order terms are negligible which holds if $\alpha^2(|\omega|^2 + k^2)^{-1} \ll 1$, noting that terms involving an odd number of derivatives vanish. This inequality relation will be checked *a posteriori*. Evaluating (10), we get

$$k^2 - av_0^2 + k^2(a+k^2) = \zeta Z(\zeta)[\tfrac{1}{2}a(1+3v_0^2) - k^2 - \zeta^2(1+v_0^2)a] - \zeta^2(1+v_0^2)a, \tag{11}$$

where $\zeta = \omega/\sqrt{2}k$, $Z(\zeta)$ is the plasma dispersion function, and the vortex is represented by the parameter $a = \frac{25}{11}\alpha^2$.

Equation (11) is a new dispersion relation derived for linear modes in an inhomogeneous plasma characterized by the presence of a vortex structure in phase space. It includes, of course, the homogeneous dispersion relation $1 - (1/2k^2)Z'(\zeta) = 0$ which comes out in the limit $a \to 0$ or $k \to \infty$. It possesses, in addition, two new branches, a marginally stable branch given by $\omega^2 = 1 + 3(2+v_0^2)ak^2/(a+k)^2$, $k^2 \ll 1$, which survives in the one-dimensional limit ($k \to 0$), and an aperiodic branch ($\mathrm{Re}\,\omega = 0$). The unstable part of the latter is depicted in Fig. 1 in which $\gamma = \mathrm{Im}\,\omega$ is plotted as a function of k for three values of the vortex parameter a. This branch represents purely growing modes for $k < k_{\mathrm{cr}} \approx k_* \equiv a^{1/2}v_0$ and purely damped modes for $k > k_{\mathrm{cr}}$. The dependence of the scaled quantities on the parameter a is weak. Maximum growth rate is attained at $k_{\max} \approx k_*/2$ with $\gamma_{\max} \approx k_*/2 \sim \alpha \sim \psi_0^{1/4}$. It scales, therefore, with the inverse width of the inhomogeneity which means that the growth rate increases with an increasing nonlinearity. The same is true for $k_{\max}$ indicating among others that the results are not available by a WKB analysis. The figure also shows that γ, and with it the branch, disappears in the one-dimensional limit, $k \to 0$. Hence, electron phase-space vortices are stable in one dimension but unstable in two and three dimensions.

With regard to the validity of the present approach it should be said that the cutoff in the geometric series in (10) is satisfied only marginally for the most unstable mode in view of $\alpha^2/(\gamma_{\max}^2 + k_{\max}^2) = 0.5$. In any case, the fluid approach seems to be better justified than the kinetic limit where the opposite inequality relation is assumed. The validity and range of the second assumption, namely the representation of φ_1 by η_0 alone, can be checked by the inclusion of more than one eigenfunction of Λ in φ_1, which is an interesting extension of the present work and in progress.

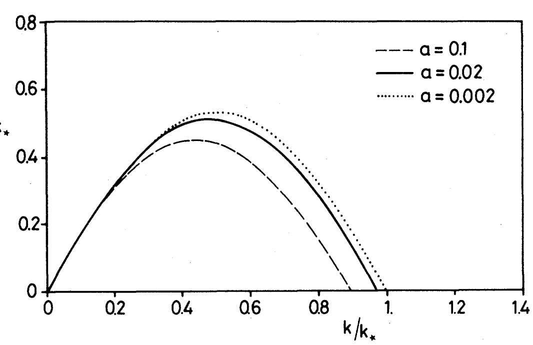

FIG. 1. The growth rate as a function of wave number for different values of the vortex parameter a.

I close with the remark that these results explain for the first time analytically a well-known property of vortices, namely, the major qualitative differences which exist between one and two dimensions and are seen in numerical simulations,[1] where it was found that the persistence of vortex structures is lost in going from one to two (or three) dimensions.

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