Since the metric $g_{ij} + f_i f_j$ is asymptotically flat, we can apply our previous arguments in Ref. 11 to prove that the metric $g_{ij} + f_i f_j$ can be deformed conformally to an asymptotically flat metric whose scalar curvature is zero and whose ADM mass is equal to $2p$ minus a nonnegative quantity. Hence by the theorem that we proved in Ref. 1, $2p$ is positive. Therefore, it remains to prove that f exists with the required asymptotic expansion where p is a positive multiple of the Bondi mass.

The existence of the solution f to Eq. (1) with the required expansion can be proved in the same way as in Ref. 11. We take a sequence of spheres S_i tending to infinity of M and on each S_i, we define f_i so that it behaves like $r + p \ln r + q r^{-1}$ on S_i. Then we try to solve the boundary-value problem for Eq. (1) with boundary value f_i. This is done exactly as in Ref. 11 by perturbing of Eq. (1) and taking a limit. The boundary-value problem can be solved in the generalized sense by allowing f_i to go to infinity on the apparent horizon of M. A generalization of Eq. (2) is crucial in the proof. This is Eq. (2.25) in Ref. 11. After solving the boundary-value problem, we let i tend to infinity and prove that f_i converges to a solution f with the required asymptotic expansion. As in Ref. 11, the behavior of f near the apparent horizon is well understood. As a result, we can assume that f does not blow up for all practical purposes (see the arguments in Sec. 4 of Ref. 11).

Once the existence of f has been established, the relation of p (in the expansion of f) to the Bondi mass can be computed by using Eq. (1). It is a positive multiple of m and this finishes the proof of the positivity of the Bondi mass.

Finally, we remark that Ref. 16 can be used to take care of the possible angular dependence of m.

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Gravitational Energy Cannot Become Negative

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The positive-energy conjecture is proved at null infinity. That is, an isolated system in general relativity can never radiate away more energy than is given by its total Arnowitt-Deser-Misner energy.

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Most physical systems cannot radiate away more energy than they have initially. This is usually a trivial consequence of a conserved stress-energy tensor with a positive timelike component. However, as is well known, the gravitational field does not have a well-defined

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stress-energy tensor. Thus, there is the possibility that a finite gravitating system might be able to radiate arbitrarily large amounts of energy. The idea that this is impossible in general relativity is known as the positive-energy conjecture at null infinity.

An isolated system in general relativity is described by a space-time which is asymptotically flat. The curvature vanishes both at large spacelike distances from the source (spatial infinity) and at large null distances (null infinity). In each of these asymptotic regions one can define an asymptotic energy-momentum four-vector; at spatial infinity it is called the Arnowitt-Deser-Misner[1] four-momentum P_a^{ADM}, whereas at null infinity, it is called the Bondi[2] four-momentum P_a^{B}.

The difference between these two quantities is illustrated by Fig. 1. The vector P_a^{ADM} represents the net energy and momentum crossing an asymptotically flat spacelike surface Σ. Since Σ eventually intersects all emitted radiation, P_a^{ADM} is time independent, and represents the *total* energy of the system. However, P_a^{B} represents the net energy and momentum crossing an asymptotically null surface N (that is, a spacelike surface which asymptotically approaches a null surface of constant retarded time, u). Since N does not intersect all emitted radiation, P_a^{B} depends on u and represents the *remaining* energy-momentum of the system at time u. Not surprisingly, at future null infinity, $\mathscr{I}^+$, the past limit of P_a^{B} is equal to P_a^{ADM},[3] and the Bondi energy (the contraction of P_a^{B} with any time translation) is a decreasing function of u.[2]

It has recently been shown that if the stress-energy tensor obeys a local positivity condition,

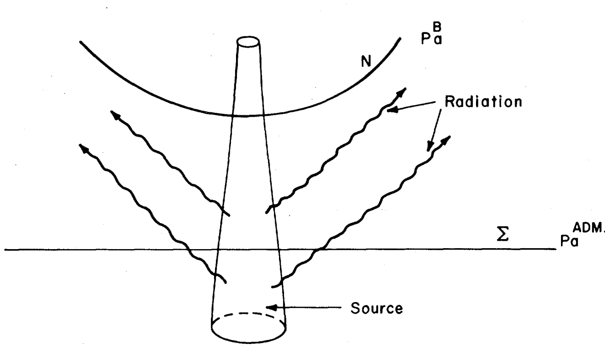

FIG. 1. The ADM energy-momentum, as measured on surfaces like Σ, is independent of time even though the source is evolving. The Bondi energy-momentum as measured on surfaces like N is time dependent.

then P_a^{ADM} is a future-directed timelike or null vector. This was first established by Schoen and Yau,[4] and later given a simple proof by Witten.[5] Thus, physically reasonable gravitating systems initially have positive total energy. The question of whether such systems can radiate more energy than they initially have is the question of whether the Bondi energy can become negative. We shall show here that it cannot. To be precise, we prove the following.

Theorem.—Let (M, g_{ab}) be a space-time that satisfies the following: (i) It is asymptotically flat at future null infinity.[6] (ii) The dominant energy condition holds[7]: $T_{ab}t^a$ is future directed for all future-directed t^a. (iii) There exists a nonsingular spacelike surface N which asymptotically approaches the null cone of constant retarded time u near null infinity. Then $P_a^{\mathrm{B}}(u)$ is a future-directed timelike or null vector. Furthermore, $P_a^{\mathrm{B}}(u)$ vanishes iff the space-time is flat in the domain of dependence of N, $D(N)$.

Recall that $D(N)$ (Ref. 7) is the region of M which is determined by initial data on N. Since N is asymptotically null, this is not the entire space-time. Note that although the statement of this theorem is similar to that about P_a^{ADM}, the present theorem is much stronger: If P_a^{B} is future directed, then P_a^{ADM} is also future directed. A similar theorem holds for past null infinity.

The technique to be used in proving the theorem is similar to that used by Witten,[5] and involves a recently discovered generalization[8] of Witten's proof. Schoen and Yau have independently found a proof of the above theorem[9] using methods similar to their previous work.[4] We will use two-component spinor notation,[10] with spinor indices denoted by capital letters and tensor indices (each equivalent to a pair of spinor indices) by lower-case letters.

Let (M, g_{ab}) be a space-time which satisfies the three conditions of the theorem. Let α^A be a solution of the Weyl equation

$$\nabla_{AA'}\alpha^A = 0. \tag{1}$$

We now calculate $\nabla^b\nabla_{[a}K_{b]}$ where $K_b = \alpha_B\bar{\alpha}_{B'}$ is the null vector determined by α_B. Expansion of the antisymmetrization yields two terms. For the first, commute the derivatives and note that K_b is divergence free from (1). For the second, recall that (1) implies that

$$\nabla^m\nabla_m\alpha^A + \tfrac{1}{4}R\alpha^A = 0. \tag{2}$$

Combination of these terms yields

$$\nabla^b\nabla_{[a}K_{b]} = -\nabla_b\alpha_A\nabla^b\bar{\alpha}_{A'} + 4\pi T_{ab}K^b, \tag{3}$$

where Einstein's equation has been used to replace the curvature terms by the stress-energy tensor. Integration of this over N yields

$$\tfrac{1}{2}\int_s \nabla_a K_b\, dS^{ab} = \int_N (-\nabla_b\alpha_A\nabla^b\bar{\alpha}_{A'} + 4\pi T_{ab}K^b)\,d\Sigma^a. \tag{4}$$

S is a two-sphere at null infinity with volume element dS^{ab} normal to S. This equation plays an important role in our proof of the theorem.

Our proof of the first part of the theorem proceeds in two steps. First, we show that α^A can be chosen to have asymptotic behavior so that the left-hand side of (4) is related to $P_a^{\mathrm{B}}(u)$. Then, we show that this α^A can be extended to a smooth Weyl solution in a neighborhood of N with the right-hand side of (4) positive.

For the first step we briefly review the Geroch-Winicour formulation of asymptotic linkages.[11] Let ξ^a be a divergence-free vector field which asymptotically becomes a generator of a spacetime symmetry. To be more precise, one requires that ξ^a admit a smooth extension to null infinity such that its restriction to $\mathscr{I}^+$, $\xi^{(0)a}$, is a generator of the Bondi-Metzner-Sachs group.[2,12] The linkage associated with ξ^a at retarded time u is

$$L(\xi) = (8\pi)^{-1}\int_s \nabla_a\xi_b\, dS^{ab}, \tag{5}$$

where S is an asymptotic two-sphere at given retarded time. If ξ^a is asymptotically a translation $\xi^{(0)a}$, then the linkage defines the Bondi four-momentum; $L(\xi) = \xi^{(0)a}P_a^{\mathrm{B}}(u)$. Now, the left-hand side of (4) is identical to (5), and it is an immediate consequence of the Weyl equation that K^a is divergence free. Therefore, one need only choose α^A such that K^a becomes asymptotically a null translation, to have the left-hand side of (4) related to $P_a^{\mathrm{B}}(u)$. This is always possible,[8] and such solutions to the Weyl equation will be called *asymptotically constant.*[13]

We now turn to the second step. Consider the right-hand side of (4). The stress-energy term always contributes positively by virtue of the dominant energy condition. The first term can in general have either sign. However, suppose there exists a unit timelike vector field $\hat{t}^a$ such that on N

$$\hat{t}^m\nabla_m\alpha_A = 0. \tag{6}$$

Since the metric orthogonal to $\hat{t}^a$ is negative definite, the first term is now positive. We thus consider solutions of the Weyl equation that obey (6) on N. One can characterize such solutions by their initial data. Recall[14] that given any spinor field α^A on a spacelike surface Σ, there exists a unique solution of the Weyl equation in $D(\Sigma)$ which agrees with the initial data α^A.

We now wish to choose initial data such that (6) is satisfied on N. We decompose $\hat{t}^a$ into components normal and tangential to N:

$$f\hat{t}^a = t^a + v^a, \tag{7}$$

where t^a is the unit normal to N, $t^a v_a = 0$, and $f = (1 + v^a v_a)^{1/2}$. Equation (6) together with the Weyl equation is equivalent to the following equation on N:

$$D_{AA'}\alpha^A - t_{AA'}v^b D_b\alpha^A = 0, \tag{8}$$

where $D_a = \nabla_a - t_a t^b\nabla_b$ is the projection of ∇_a into N. If $v^a = 0$, this equation reduces to the equation Witten used to prove the positivity of the ADM mass. However, this choice is not as useful on asymptotically null surfaces, since it is difficult to prove the existence of solutions. If one writes $D_{AA'}\alpha^A = 0$ in terms of ordinary derivatives in N, one finds that there are coefficients which become unbounded asymptotically and standard existence theorems break down. However, we can now use the freedom in the choice of $\hat{t}^a$ to find an equation with well-behaved asymptotic behavior. For any choice of v^a with $|v^a v_a| < 1$, Eq. (8) is uniformly elliptic, and one can show from the positivity of Eq. (4) that (8) has vanishing kernel.[15] We claim that if $\hat{t}^a$ is chosen to be asymptotically a *time translation* (i.e., $\hat{t}^a$ admits an extension such that its restriction to $\mathscr{I}^+$ is a BMS time translation) then (8) has the correct asymptotic behavior such that standard theorems can be applied to prove existence of asymptotically constant solutions to (8). To see this, let $\alpha^{(0)A}$ be an arbitrary asymptotically constant spinor on N, and $K^{(0)A} = \alpha^{(0)A}\bar{\alpha}^{(0)A'}$. Denoting the differential operator in (8) by L, we wish to solve

$$L\alpha^{(1)} = -L\alpha^{(0)} \tag{9}$$

with $\alpha^{(1)} = O(r^{-1})$. Expanding (9) in an asymptotically Cartesian coordinate system[16] one finds that the coefficients of the first-order derivatives tend to constants, and the coefficients of the nonderivative terms vanish faster than r^{-1}. Thus one can apply the theorem of Christodoulou and Choquet-Bruhat[17] to conclude that solutions exist.

This shows that $K^{(0)a}P_a{}^B(u) \geq 0$. Since $K^{(0)a}$ is an arbitrary null vector, we conclude that $P_a{}^B(u)$ must be future directed. This completes the proof of the first part of the theorem.

This result could have been obtained by working entirely with fields on N and not using the Weyl equation. However, the calculations are rather more complex. We sketch this alternative. Again we start with an asymptotically constant solution to (8). Any such solution obeys the identity

$$-D^a[t^b D_a K_b + h_a{}^b v^c D_c K_b] = -2t^{AA'}[D_b \alpha_A D^b \overline{\alpha}_{A'} + (v^b D_b \alpha_A)(v^c D_c \overline{\alpha}_{A'})] + 8\pi T_{ab} t^a K^b, \tag{10}$$

where $h_a{}^b = \delta_a{}^b - t_a t^b$ is the projection operator into N. The right-hand side of this identity is nonnegative since $|v^a v_a| < 1$. One integrates (10) over N, and converts the left-hand side into a surface integral. Finally, it follows from (8) that this surface integral is just $8\pi K^{(0)a}P_a{}^B(u)$.

To prove the second part of the theorem we use an argument which is essentially identical to one used by Witten.[5] Assume $P_a{}^B(u) = 0$. As before, let α^A be any asymptotically constant solution to (8) on N, and evolve to obtain a solution to the Weyl equation in a neighborhood of N. Since the left-hand side of (4) vanishes, the right-hand side must vanish. This implies $D_m \alpha^A = 0$ on N. Since there exists a basis of covariantly constant spinors on N, we have

$$h_a{}^m h_b{}^n R_{mnpq} = 0. \tag{11}$$

By making local deformations of N and repeating this argument we conclude $R_{abcd} = 0$ in $D(N)$. This completes the proof.

Another proof of the first part of this theorem has recently been given by Ludvigsen and Vickers.[18]

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[16] Let u, r, θ, and φ be Bondi-Sachs asymptotic coordinates. We define new coordinates by $t = u + r$, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. In Bondi-Sachs coordinates, an expansion of Eq. (8) into its components contains nonderivative terms which are $O(r^{-1})$, arising from the fact that the induced spinor basis is not constant, even in flat space-time. In the asymptotically Cartesian coordinate system (t,x,y,z), such terms are not present.

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