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Proof That the Bondi Mass is Positive

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It is demonstrated that the total mass of a nontrivial isolated physical system is positive even after part of its mass has been lost because of gravitational radiation. This method is an extension of the one that the authors used in the proof of the classical positive-mass conjecture.

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The positive-mass theorem states that for a nontrivial isolated physical system, the total energy, which includes contributions from both matter and gravitation, is positive. This theorem was first proved by the authors.[1,2] Recently, Witten[3] has published an alternative proof depending on the existence of a certain harmonic spinor, which is asymptotic to a constant spinor in a certain rate. Very recently, Parker and Taubes[4] were able to use some estimates of Nirenberg and Walker[5] and Cantor[6] to justify the existence with the required rate. (Witten has pointed out that his idea came from supergravity. In fact, Grisaru,[7] following the work of Deser and Teitelboim,[8] has given a formal proof of the positivity using supergravity.)

After the confirmation of the total positivity of the total mass (as viewed from the spatial infinity), it is natural to ask whether the mass is still positive if part of the mass has been lost as a result of gravitational radiation. The theory of gravitational radiation was worked out by Bondi, Van der Burg, and Metzner[9] and Sachs.[10] They associate to each null cone a number which is called the Bondi mass of the null cone. The Bondi mass is interpreted as the total mass of the isolated physical system measured after the loss due to the gravitational radiation up to that time. Bondi and Sachs proved that when time goes on, the Bondi mass decreases and the loss is measured by the "news function."

The purpose of this Letter is to show that our previous argument[2] can be modified to demonstrate the positivity of the Bondi mass. Our idea can be described as follows. Given a spacelike hypersurface asymptotic to the given null cone in the space-time, we construct a three-dimensional asymptotically flat initial-data set whose Arnowitt-Deser-Misner (ADM) mass is not greater than the Bondi mass of the given null cone. The positivity of the Bondi mass then follows from our previous proven theorem on the positivity of the ADM mass. Note that this new initial-data set is not necessarily embeddable in the original space-time. The construction of this new initial-data set is very similar to the construction that we used earlier.[11] Our construction allows the existence of singularities of the black-hole type. It will be described in more detail in what follows.

It should be noted that Horowitz and Perry[12] have independently announced a proof of the positivity of the Bondi mass under the assumption that the space-time is asymptotically flat in the sense of Penrose[13] and that there exists a nonsingular three-dimensional spacelike hypersurface which is asymptotic to the null cone. Their method is very different from ours. They modify Witten's approach. While our method is very sensitive to the choice of the spacelike hypersurface asymptotic to the null cone, their method is not. We were also informed that Israel and Nester[14] and Ludvigsen and Vickers[15] also announced a similar result recently.

We follow Sachs's paper and write the space-time metric in a normal form. Thus, let u be a retarded coordinate, r be the radial coordinate, and θ and φ be the spherical coordinates. Then we assume that the metric can be written as

$$(Ve^{2\beta}/r)du^2 - 2e^{2\beta}du\,dr + r^2 h_{AB}(dx^A - u^A du)(dx^B - u^B du),$$

where

$$2h_{AB}dx^A dx^B = (e^{2\gamma} + e^{2\delta})d\theta^2 + 4\sin\theta\sinh(\gamma-\delta)d\theta\,d\varphi + (\sin\theta)^2(e^{-2\gamma} + e^{-2\delta})d\varphi^2.$$

(Here the indices A and B range from 2 to 3 and $x^2 = \theta$, $x^3 = \varphi$.)

We assume that

$$V = -r + m + O(r^{-1}),$$

$$\beta = -|c|^2/4r^2 + O(r^{-4}),$$

$$\gamma = (\mathrm{Re}\,c + \mathrm{Im}\,c)/r + O(r^{-3}),$$

$$\delta = (\mathrm{Re}\,c - \mathrm{Im}\,c)/r + O(r^{-3}).$$

The Bondi mass for the slice $u=0$ is defined by integrating m along the sphere at infinity.

By applying a supertranslation, we can transform the null space $u=0$ to a future null space $u=\alpha$ with $\alpha \geq 0$ so that with respect to the new coordinate system $u' = u - \alpha$, the function c is purely imaginary. We are going to prove that the Bondi mass associated with the null cone $u'=0$ is positive. By the mass-loss formula of Bondi, this will prove the positivity of the Bondi mass of the null cone $u=0$. As a consequence, we can assume, without loss of generality, that c is purely imaginary.

Our idea now is to choose a spacelike hypersurface H of the form $u = r^{-1} + ar^{-3} + br^{-4} + O(r^{-5})$ so that the radial part of the induced metric on this hypersurface behaves like $[1 + mr^{-3} + O(r^{-4})](1 + r^2)^{-1}(dr)^2$. By choosing the coefficient of r^{-4} suitably in the above expansion of u, we can arrange that the radial term of the second fundamental form of H behaves like $[1 + O(r^{-4})](1+r^2)^{-1}dr^2$.

With this choice of H, we are able to find a new initial-data set so that the total mass of this initial-data set is a positive multiple of the Bondi mass of the null cone $u=0$. The construction of this initial-data set is very similar to our method of proving the positivity of the ADM mass and can be described as follows.

As in Ref. 11, we seek a function f which solves the equation

$$\sum_{i,j}\left(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\right)\left(\frac{f_{ij}}{(1+|\nabla f|^2)^{1/2}} - p_{ij}\right) = 0, \quad (1)$$

where g_{ij} is the induced metric on H, p_{ij} is the second fundamental form of H, and f_{ij} is the second covariant derivative of f in directions x^i and x^j.

We replace the metric g_{ij} on H by the metric $g_{ij} + f_i f_j$. In order for the new metric $g_{ij} + f_i f_j$ to be asymptotically Euclidean, we require f to be asymptotically $r + p\ln r + qr^{-1} + \ldots$. It is clear that such an asymptotic expansion of f guarantees that the metric $g_{ij} + f_i f_j$ is asymptotically flat and its ADM mass is $2p$.

In our previous papers (Refs. 2 and 11), we have already demonstrated that if Eq. (1) holds, then the dominant energy condition can be formulated in a more geometric fashion. Namely, let $\bar{H}$ be the graph of f in the product space $M \times R$. Then let $\bar{R}$ be the scalar curvature of the induced metric (which is given by $g_{ij} + f_i f_j$) and let h_{ij} be the second fundamental form of $\bar{H}$ in the space $M \times R$. Then

$$\bar{R} \geq \sum (h_{ij} - p_{ij})^2 + 2\sum_i (h_{i4} - p_{i4})^2 - 2\sum_i \bar{D}_1 (h_{i4} - p_{i4}), \quad (2)$$

where the index 4 corresponds to the normal of the graph of f and $\bar{D}_i$ is the covariant differentiation taken on the graph.

Since the metric $g_{ij}+f_if_j$ is asymptotically flat, we can apply our previous arguments in Ref. 11 to prove that the metric $g_{ij}+f_if_j$ can be deformed conformally to an asymptotically flat metric whose scalar curvature is zero and whose ADM mass is equal to $2p$ minus a nonnegative quantity. Hence by the theorem that we proved in Ref. 1, $2p$ is positive. Therefore, it remains to prove that f exists with the required asymptotic expansion where p is a positive multiple of the Bondi mass.

The existence of the solution f to Eq. (1) with the required expansion can be proved in the same way as in Ref. 11. We take a sequence of spheres S_i tending to infinity of M and on each S_i, we define f_i so that it behaves like $r+p\ln r+qr^{-1}$ on S_i. Then we try to solve the boundary-value problem for Eq. (1) with boundary value f_i. This is done exactly as in Ref. 11 by perturbing of Eq. (1) and taking a limit. The boundary-value problem can be solved in the generalized sense by allowing f_i to go to infinity on the apparent horizon of M. A generalization of Eq. (2) is crucial in the proof. This is Eq. (2.25) in Ref. 11. After solving the boundary-value problem, we let i tend to infinity and prove that f_i converges to a solution f with the required asymptotic expansion. As in Ref. 11, the behavior of f near the apparent horizon is well understood. As a result, we can assume that f does not blow up for all practical purposes (see the arguments in Sec. 4 of Ref. 11).

Once the existence of f has been established, the relation of p (in the expansion of f) to the Bondi mass can be computed by using Eq. (1). It is a positive multiple of m and this finishes the proof of the positivity of the Bondi mass.

Finally, we remark that Ref. 16 can be used to take care of the possible angular dependence of m.

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Gravitational Energy Cannot Become Negative

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The positive-energy conjecture is proved at null infinity. That is, an isolated system in general relativity can never radiate away more energy than is given by its total Arnowitt-Deser-Misner energy.

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Most physical systems cannot radiate away more energy than they have initially. This is usually a trivial consequence of a conserved stress-energy tensor with a positive timelike component. However, as is well known, the gravitational field does not have a well-defined

