Agreement of Capillary-Wave Theory with Exact Results for the Interface Profile of the Two-Dimensional Ising Model

Abraham[1] has recently calculated several properties of interfaces in two-dimensional (2D) Ising models, in particular the interface profile as the system size tends to infinity. He observes that his exact results take the functional forms predicted by simple capillary-wave theory[2] but with coefficients involving the interface tension in disagreement with the capillary-wave predictions. However, the usual capillary-wave theory[2] assumes an isotropic fluid with an interface tension independent of angle. This assumption does not apply to the Ising model at low temperatures for which the interface tension has a significant angular dependence.[3,4]

In this note we point out that if generalized capillary-wave expressions are used[5] with the (possibly angular dependent) interface tension defined in the proper macroscopic manner, then the coefficients entering Abraham's expressions for the interface profile are also given exactly by the generalized capillary-wave theory at all temperatures. This exact result thus provides additional support for the physical picture of interface wandering given by the capillary-wave theory.[2,5]

In a lattice system such as the 2D Ising model, the interface tension $\gamma(\theta)$, defined as the interfacial free energy per unit *Cartesian* length for a wall with average angle θ, is at low temperatures generally anisotropic.[3,4] [As $T \to T_c$, $\gamma(\theta)$ becomes independent of θ, as is the case for an ordinary fluid.[4]] There will be a minimum at an angle $\theta = 0$ (i.e., parallel to a lattice axis), and for small angles we can expand $\gamma(\theta) = \gamma(0) + \frac{1}{2}\gamma''(0)\theta^2 + \dots$. There is no linear term in θ at any nonzero temperature since the roughening temperature for the 2D Ising model is zero.[3,6] If we have a macroscopic system of width L and impose a tilt with average angle θ on a horizontal interface the new length of the interface is $L \sec\theta$ and the change in free energy due to this distortion is

$$\Delta F = \gamma(\theta) L \sec\theta - \gamma(0) L = \tfrac{1}{2} L \theta^2 [\gamma(0) + \gamma''(0)] \tag{1}$$

for small θ. The resistance to small distortions, which forms the basis for capillary-wave theory,[2] is thus controlled by an "effective" interface tension[5]

$$\Gamma = \gamma(0) + \gamma''(0) \tag{2}$$

rather than $\gamma(0)$ alone. At very low temperatures $\gamma''(0)$ diverges exponentially[5] while $\gamma(0)$ approaches a constant, and hence the correction plays an essential role.

Abraham[1] proves that the second moment of the interface profile for an isotropic Ising model at temperature $T < T_c$ in the limit $L \to \infty$ is given exactly by

$$w^2 = L / \sinh[\gamma(0)/T], \tag{3}$$

while the modified capillary-wave theory[5] predicts $w^2 = LT/\Gamma$. [This is just the usual result[1,2] with $\gamma(0)$ replaced by Γ as discussed above].

Abraham and Reed[4] have calculated exact expressions for the interface tension as a function of angle for the 2D Ising model. [The results of Ref. 4 should be multiplied by $(1 + |\tan\theta|)\cos\theta$ to obtain $\gamma(\theta)$ as it is defined herein.] Using these we find the exact result

$$\sinh[\gamma(0)/T] = T^{-1}[\gamma(0) + \gamma''(0)]. \tag{4}$$

This shows that the generalized capillary-wave prediction is in precise agreement with the exact result (3) for all $T < T_c$ for the 2D Ising model. Other aspects of the profile (e.g., the error function shape) are also given exactly by the generalized capillary-wave theory. See Ref. 5 for further discussion of these points and of the general behavior of interfaces in 2D systems.

We are grateful to D. B. Abraham for informative discussions.

Matthew P. A. Fisher[(a)]
Daniel S. Fisher
John D. Weeks
Bell Laboratories
Murray Hill, New Jersey 07974

Received 16 November 1981
PACS numbers: 68.10.Cr, 05.50.+q, 75.10.Hk

[(a)]Present address: Department of Physics, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

[1]D. B. Abraham, Phys. Rev. Lett. 47, 545 (1981). See also D. B. Abraham and P. Reed, Phys. Rev. Lett. 33, 377 (1974), and Commun. Math. Phys. 49, 35 (1975).

[2]F. P. Buff, R. A. Lovett, and F. H. Stillinger, Phys. Rev. Lett. 15, 621 (1965); J. D. Weeks, J. Chem. Phys. 67, 3106 (1977).

[3]C. Herring, Phys. Rev. 82, 87 (1951); C. Herring, in *Structure and Properties of Solid Surfaces*, edited by R. Gomer and C. S. Smith (Univ. of Chicago Press, Chicago, 1953), p. 5.

[4]D. A. Abraham and P. Reed, J. Phys. A 10, L121 (1977).

[5]M. P. A. Fisher and J. D. Weeks, to be published. See also Ref. 3.

[6]See, e.g., H. J. Leamy, G. H. Gilmer, and K. A. Jackson, *Surface Physics of Materials* (Academic, New York, 1975), Vol. 1, p. 121.

