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Calculation of Newton's Gravitational Constant in Infrared-Stable Yang-Mills Theories

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Newton's gravitational constant G is calculated in a class of scale-invariant gauge theories with an infrared fixed point. The sign of G depends on the coefficients in the renormalization-group β function.

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Is Newton's gravitational constant G a fundamental parameter or is it calculable in terms of other fundamental parameters? In this paper I would like to argue the latter view and to present a calculation of G, unfortunately not in the real world, but in a toy world, just to demonstrate that G is indeed calculable.

The form of the non-Abelian gauge field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ dictates that the gauge potential A_μ must have dimension one regardless of the dimension of space-time, and so the Yang-Mills action F^2 must always have dimension four. (It is tempting to suggest that this fact may be connected to the actually observed dimension of space-time.) In contrast, the Einstein-Hilbert action R, being just the scalar curvature, always has dimension two. In this sense, Yang-Mills theory is matched perfectly to the observed four-dimensional space-time while gravity is not. More precisely, if we demand the fundamental theory of the world to be scale invariant, Einstein's theory is excluded. (Furthermore, in a gauge-invariant theory without any fundamental scalar fields, all terms proportional to R such as $\varphi^2 R$ are also excluded.)

It is extremely attractive to impose scale invariance since in a scale-invariant theory with n dimensionless couplings all dimensionless ratios of dimensional physical parameters are calculable[1] in terms of $n-1$ dimensionless couplings. (Some physicists harbor the ultimate ambition that n will eventually be reduced to 1.) Newton's gravitational constant G would then be calculable in terms of a purely flat-space quantity determined by the other interactions. In this context, a formula for G was derived independently by Adler[2] and Zee[3] and reads

$$(16\pi G_{\text{ind}})^{-1} = (i/96)\int d^4x\, x^2 \psi(-x^2) \qquad (1)$$

(the subscript "ind" denotes "induced") with $\psi(-x^2) \equiv \langle \mathcal{T} T(x) T(0)\rangle_0 - \langle T\rangle_0^2$. This formula expresses G_{ind} in terms of a space-time integral over the vacuum value of the time-ordered product of the trace of the stress-energy tensor $T(x)$.

The philosophy and the physics behind the derivation of Eq. (1) have been amply discussed in the literature[3-7] and will not be elaborated here. If this philosophy is correct, we would be in the exciting position of being able to understand the sign of the gravitational constant.[6] The magnitude of G is merely set by the scale of dynamical scale-invariance breaking.

Actually, the formula in Eq. (1) holds only when the metric is not itself quantized; otherwise, there are extra terms due to fluctuations in the metric which have been worked out by Adler.[5] With the metric quantized, the scale-invariant fundamental action of gravity would consist of a linear combination of R^2, $R_{\mu\nu}^2$, and $R_{\mu\nu\lambda\rho}^2$.

In this paper, I treat, for simplicity's sake, the background metric as classical and content myself with studying the formula in Eq. (1). I must mention that this formula is defined only with the understanding that it is to be evaluated with the aid of dimensional regularization. By dimensional considerations, one can see that the expression in Eq. (1) has a quadratic short-distance divergence which is prescribed to be zero by dimensional regularization. After performing a Wick rotation to Euclidean space we write Eq.

(1) as

$$(16\pi G_{\rm ind})^{-1} = -\tfrac{1}{96}\int_E d^4x\, x^2 \psi(x^2). \qquad (2)$$

If the strong, electromagnetic, and weak interactions are described by a grand unified gauge theory with massless fermions, the operator $T(x)$ is determined via the trace anomaly[8] to be

$$T = [2\beta(g)/g]\times \tfrac{1}{4} F_{\mu\nu}{}^a F^{a\mu\nu}. \qquad (3)$$

A crude calculation of $G_{\rm ind}$ taking into account the short-distance ultraviolet region was given in Ref. 3 and motivated the derivation of Eq. (1). Unfortunately, we were led to conclude that the sign of $G_{\rm ind}$ depends on the long-distance infrared region, of which we are totally ignorant. There was also a calculation[9] of G using a dilute instanton-gas approximation, but again, because of our ignorance of the long-distance physics, the infrared region was excluded by an artificial cutoff on the instanton size. Thus, neither of these calculations is conclusive as regards the sign of $G_{\rm ind}$. To include the infrared region, Adler has outlined[5] a program based on numerical lattice calculations.

In this paper, I remark that there is a class of gauge theories in which the infrared region is completely known—in fact, the function ψ may be computed explicitly. These are gauge theories with the property that in the expansion of the renormalization-group function $\beta(g) = -\frac{1}{2}g^3(b_0 + b_1 g^2 + \ldots)$ the coefficient b_0 is positive and small (so that the theory is barely asymptotically free) while the coefficient b_1 is negative. (Such theories are well known to exist; quantum chromodynamics with sixteen quark triplets provides an example.[10]) There is then an infrared-stable fixed point given by $g_*{}^2 = -b_0/b_1$. By choosing appropriately the gauge group and the fermion representations, we can make $g_*{}^2$ arbitrarily small. The calculation given below is "exact" to the extent that $g_*{}^2$ is small.

I define a dimensionless distance scale parameter $\tau \equiv \mu^2 x^2$ with μ^2 the renormalization scale mass, and also, as usual, the running coupling constant $g(\tau)$ by

$$d\tau/\tau = -2\,dg(\tau)/\beta(g(\tau)). \qquad (4)$$

It is convenient to introduce the notation $\alpha \equiv g^2$, $\alpha^* = g_*{}^2$, $y \equiv \alpha(\tau)/\alpha^*$, and $\gamma \equiv 2/b_0\alpha^*$. For α^* sufficiently small, this differential relation may be integrated "exactly" to give

$$\tau = e^{2\gamma} e^{-\gamma/y}[y/(1-y)]^{\gamma}. \qquad (5)$$

The solution is fixed by the boundary condition, chosen by convenience to be $\alpha(1) = \frac{1}{2}\alpha^*$. As is usual in a scale-invariant theory, this choice is completely arbitrary [as long as $0 < \alpha(1) < \alpha^*$]; a different choice merely shifts μ^2. As expected, y increases monotonically from 0 to 1 as τ goes from 0 to ∞. For $\tau \to 0$ we have the asymptotically free behavior

$$y \to \frac{\gamma}{\ln\tau^{-1}}\left[1 - \frac{\gamma \ln\ln\tau^{-1}}{\ln\tau^{-1}} + \ldots\right]. \qquad (6)$$

In the infrared limit $\tau \to \infty$ we have the characteristic power-law[11] approach to the fixed point

$$y = 1 - e^2 y e^{-1/y}\tau^{-1/\gamma} \to 1 - e\,\tau^{-1/\gamma}. \qquad (7)$$

For g^* small enough, the two-point function $\langle \mathcal{T} F^2(x) F^2(0)\rangle_0 - \langle F^2\rangle_0{}^2$ is well approximated by its free-field-theory value $\sim C/(x^2)^4$. The overall numerical constant C has been given by Adler.[5]

Putting all this together we find that the gravitational constant is given by

$$\frac{1}{16\pi G_{\rm ind}} = -\frac{\pi^2}{96}\left(\frac{Cb_0{}^2}{16}\right)\alpha^{*2}\mu^2 \lim_{\omega\to 2} K(\omega) \qquad (8)$$

with the dimensionless integral

$$K(\omega) \equiv \int_0^\infty d\tau \left[\frac{\tau^{\omega-1}\tau}{\tau^{2\omega}}\right][y(1-y)]^2. \qquad (9)$$

The constant C is positive. I have dimensionally continued to space-time with dimension 2ω. In Eq. (9) I have explicitly indicated how the powers of τ enter. I adopt the prescription[12] of not continuing the relation between y and τ given in Eq. (5). Dimensional regularization offers a prescription to define the integral $K(\omega)$.

I emphasize that my calculation is for a class of model field theories which presumably do not describe the real world. In particular Green's functions behave as powers rather than e^{-mx} at large distances, thus indicating the absence of a mass gap and confinement. These theories are neither confining nor symmetry breaking. In a realistic calculation we presumably would like the theory, a grand unified theory of the three other interactions, to be in the symmetry-breaking Higgs phase.

It turns out to be easier to evaluate K by integrating not over coordinate space but over the running coupling constant $y = \alpha(\tau)/\alpha^*$. Changing integration variables we find

$$K(\omega) = \gamma e^{-2\eta}\int_0^1 dy\, y^{-\eta}(1-y)^{1+\eta} e^{\eta/y}. \qquad (10)$$

Here the parameter $\eta \equiv \gamma(\omega - 1)$. It continues to γ when $\omega \to 2$. It is convenient to use an inverse coupling constant $1+w\equiv 1/y$; then

$$J(\eta)\equiv\gamma^{-1}e^{\eta}K(\omega)=\int_0^\infty dw\,\frac{w^{1+\eta}}{(1+w)^3}e^{\eta w}. \tag{11}$$

So far, we have not done anything except change variables, often enough to perhaps confuse the casual reader. But, no matter how often we change variables, the integral $K(\omega)$ is not well defined for $\omega=2$. A prescription is needed. The ultraviolet divergence in Eq. (9), of the form $\int_0(d\tau/\tau^2)(\ln\tau)^{-2}$, has been merely translated into the nasty exponential blowup in Eqs. (10) and (11). Incidentally, were it not for this ultraviolet divergence, one might have naively concluded from Eq. (10) that $G_{\rm ind}$ is always negative.[13] However, as is well known, dimensional regularization can turn apparently positive integrals into negative integrals.[14]

The function $J(\eta)$ is defined by its integral representation for $-2<\eta<0$. [The lower limit $\eta>-2$ is an infrared limit and is in some sense an artifice of dimensional continuation. Consulting Eq. (7) we see that the interaction always makes the infrared region in Eq. (9) more convergent.] Our prescription to evaluate $J(\eta)$ for $\eta=\gamma>0$ is as follows. Analytically continue $J(\eta)$ from the region $-2<\eta<0$. This is easily done. We make the change of variable $x\equiv|\eta|w$ for η negative and obtain

$$J(\eta)=(-\eta)^{1-\eta}\int_0^\infty dx\,\frac{x^{1+\eta}}{(x-\eta)^3}e^{-x}\equiv(-\eta)^{1-\eta}L(\eta). \tag{12}$$

This analytic continuation (unique, of course) shows that $J(\eta)$ is perfectly well defined for $\mathrm{Re}\eta>0$ except for a cut along the positive real axis. We define, for γ positive real,

$$J^{\rm regulated}(\gamma)\equiv\tfrac{1}{2}[J(\gamma+i\epsilon)+J(\gamma-i\epsilon)]=\mathrm{Re}J(\gamma+i\epsilon)$$

in order to obtain real physical quantities.

I follow here the prescription given by Adler.[5] It is well known that within dimensional regularization, to any finite perturbative order, one only encounters poles in the complex ω plane. Here, however, the trace anomaly incorporates effects to all orders in perturbation theory. If the function β is also continued,[12] one encounters a cut on the real axis to the right of $\omega=2$ and an infinite number of poles to the left of $\omega=2$. For a more detailed analysis of this point, see Adler.[5] I argue that, unless the general argument given by Adler[2,5] that G is finite and calculable fails for some reason, the computed value of G should be independent of the regularization scheme[15] used.

We will now compute $L(\eta=\gamma+i\epsilon,\gamma>0)$. After integrating by parts three times (since the whole calculation is only valid if γ is large we drop all surface terms) we find

$$\mathrm{Im}L(\gamma+i\epsilon)=-\tfrac{1}{2}\pi\gamma^{\gamma}e^{-\gamma}, \tag{13}$$

$$\mathrm{Re}L(\gamma+i\epsilon)=-\tfrac{1}{2}\gamma^{\gamma}\int_0^\infty dt\,\ln|t-1|\,h(t), \tag{14}$$

with

$$h(t)=t^{\gamma-2}[(\gamma^2-1)-3\gamma(\gamma+1)t+3\gamma(\gamma+1)t^2-\gamma^2t^3]e^{-\gamma t}. \tag{15}$$

This form is suitable for numerical integration.

For γ very large,[16] which is when the calculation is valid anyway, this integral could be done by steepest descent. Each term in the integrand has the form of a function, $\ln|t-1|t^{\gamma}e^{-\gamma t}$, which is sharply peaked about $t\sim1$, multiplied by t^n (with $n=-2,-1,0,1$). After a tedious integration, we find that, for large γ,

$$\mathrm{Re}L=-\tfrac{2}{3}(2\pi)^{1/2}\gamma^{\gamma-1/2}e^{-\gamma}[\ln\gamma+O(1)]. \tag{16}$$

We see that $\mathrm{Re}L$ is negligible compared to $\mathrm{Im}L$ for large γ. Incidentally, we see that the major contribution in Eq. (14) comes from neither the infrared nor the ultraviolet region.

Finally, tracing through our steps, we see that the sign of Newton's constant is given by the sign of

$$\sin(\gamma-1)\pi\,\mathrm{Im}L-\cos(\gamma-1)\pi\,\mathrm{Re}L$$

and so

$$\mathrm{sgn}G=(-1)^{[\gamma]}, \tag{17}$$

where $[x]$ is the largest integer less than or equal to x.

Incidentally, the magnitude of Newton's constant turns out to be (for $\gamma\neq$ integer)

$$\frac{1}{16\pi G\mu^2}=(-)\frac{\pi^3}{192}\left(\frac{C}{4\alpha_*^{\,2}}\right)e^{-2\gamma}\sin\gamma\pi. \tag{18}$$

Note that the scale mass μ is physical by virtue of the choice $\alpha(1)=\alpha^*/2$ and may be determined in principle by measuring Green's functions in this toy world. In the real world, we must do an independent calculation of some other physical quantity, such as the proton mass or the string constant, in terms of μ.

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[10]W. Caswell, Phys. Rev. Lett. 33, 244 (1974).

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[12]In 2ω-dimensional space the function $\beta(g)$ has an additional term $(\omega-2)g$ linear in g. We encounter poles in the ω plane. For a discussion on the prescription independence of certain integrals, see Adler, Ref. 5, Appendix B.

[13]This is related to an erroneous argument on the sign of G by writing an unsubtracted dispersion relation for $\psi(t)$ as analyzed in the second paper cited in Ref. 2.

[14]A simple example is $f(\omega)=\int_0^1 d\tau\, \tau^{-\omega}$. While this integral is certainly undefined for $\omega>1$ we could analytically continue $f(\omega)$ into the complex ω plane to give $f(\omega)=(1-\omega)^{-1}$ by doing the integral for sufficiently negative ω. This then yields $f(\omega)$ negative for $\omega>1$.

[15]Alternatively, one can add $M^{-2}(DF)^2+M'^{-4}(DDF)^2$ to the Yang-Mills action with M, M' large masses [B. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); A. Slavnov, Nucl. Phys. B31, 301 (1971)]. I expect that my calculation could also be performed with this regularization.

[16]For SU(3) with sixteen fermion triplets, $g_*^2/16\pi^2 = 1/302$ and $\gamma = 906$. For SO(2) with Weyl fermions in the spinor representation, $g_*^2/16\pi^2 = 35/4812$ and $\gamma = 7218/1225 = 5.89$.