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Hidden Variables, Joint Probability, and the Bell Inequalities

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It is shown that the following statements about a quantum correlation experiment are mutually equivalent. (1) There is a deterministic hidden-variables model for the experiment. (2) There is a factorizable, stochastic model. (3) There is one joint distribution for all observables of the experiment, returning the experimental probabilities. (4) There are well-defined, compatible joint distributions for all pairs and triples of commuting and noncommuting observables. (5) The Bell inequalities hold.

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I consider correlation experiments (for instance, on pairs of dissociation fragments of a metastable molecule[1]) of the general type described by Clauser and Horne.[2] This involves distinct measurements in space-time regions R_1 and R_2 (that, ideally, would be spacelike separated), measuring noncommuting observables A, A' in R_1 and B, B' in R_2 (for instance, measuring spin components along skew directions in the plane transverse to the "path" of the particles). These observables are two valued, say with values ±1. The *probabilities of the experiment* are the observed distributions for each of the four observables, plus the observed distributions for each of the four compatible pairs: AB, AB', $A'B$, and $A'B'$. We can write these as $P(A)$, $P(\bar{B})$, $P(AB)$, $P(\bar{A}'B')$, etc.; where $P(\cdots)$ denotes the probability that the enclosed observables each take the value +1, and where the "complement" of an observable $\bar{S}$ takes the value +1 if the observable S takes the value −1.

A *deterministic hidden-variables model* for such an experiment consists of a set Λ of hidden variables (the "complete" state specifications), a normalized probability density $\rho(\lambda)$ defined on Λ, and response functions (giving the λ-determined responses to the measurement) $A(\lambda)$, $A'(\lambda)$, $B(\lambda)$, and $B'(\lambda)$, each defined on Λ with values ±1, and satisfying (where the integration is over Λ),

$$P(S) = \int \tilde{S}(\lambda)\rho(\lambda)d\lambda\,, \tag{1}$$

and

$$P(ST) = \int \tilde{S}(\lambda)\tilde{T}(\lambda)\rho(\lambda)d\lambda\,, \tag{2}$$

where $\tilde{S}(\lambda)=1$ if $S(\lambda)=1$, and $\tilde{S}(\lambda)=0$ if $S(\lambda)=-1$; similarly for $\tilde{T}$. {(In (1) S ranges over the four observables of the experiment plus their complements; and when, for example, we put $\bar{A}$ on the left-hand side, then we put $[1-\tilde{A}(\lambda)]$ on the right-hand side. In (2) the pairs ST range over the compatible pairs of observables and their complements, under the same convention.}

The first result here is to show that the existence of a deterministic hidden-variables model is strictly equivalent to the existence of a joint probability distribution function $P(AA'BB')$ for the four observables of the experiment, *one that returns the probabilities of the experiment as marginals*. To see this, note that, if we have a deterministic hidden-variables model, then the equation

$$P(AA'BB') = \int_\Lambda \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \quad (3)$$

defines a joint distribution for the four observables {again where $\bar{S}$ on the left-hand side goes along with $[1-\tilde{S}(\lambda)]$ on the right-hand side}. It then follows from (1) and (2) that the marginals yield the probabilities of the experiment. For example, the marginals $P(AA'BB')+P(A\bar{A}'BB')+P(AA'B\bar{B}')+P(A\bar{A}'B\bar{B}')$ produce $\int_\Lambda A(\lambda)B(\lambda)\times\rho(\lambda)d\lambda = P(AB)$, by (2). Conversely, if we are given a distribution $P(AA'BB')$ for the four observables, one whose marginals return the single and double experimental probabilities, then there is a simple, canonical way to define a deterministic hidden-variables model. Namely, let Λ consist of all sixteen quadruples $\lambda = \langle a_1, a_2, a_3, a_4\rangle$, where $a_i = \pm 1$. Introduce the response functions as $A(\lambda) = a_1$, $A'(\lambda) = a_2$, $B(\lambda) = a_3$, and $B'(\lambda) = a_4$. Then define the density $\rho(\lambda)$ as follows: $\rho(a_1, a_2, a_3, a_4) = P(A_1A_2'B_3B_4')$, where we write $S_i = S$ if $a_i = 1$, and $S_i = \bar{S}$ if $a_i = -1$. One can readily verify that ρ is normalized (all sixteen terms sum to 1) and that (1) and (2) hold [because the marginals of $P(\cdots)$ were assumed to yield the probabilities of the experiment].

Thus the idea of deterministic hidden variables is just the idea of a suitable joint probability function.[3] If there were such hidden variables (i.e., if there were such a joint distribution) then there would be distributions for triples of observables A, B, B' satisfying $P(ABB') = \int \tilde{A}(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\times\rho(\lambda)d\lambda$ that return the correct experimental probabilities, $P(A)$, $P(B)$, $P(B')$, $P(AB)$, and $P(AB')$, as marginals. Similarly, there would be a distribution $P(A'BB')$ for the triple A', B, B' also returning the corresponding experimental probabilities. Moreover, each of these triple distributions would give rise to one and the same distribution for the noncommuting pair B, B' as $P(BB') = \int \tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda$. Thus, inevitably, the existence of deterministic hidden variables violates the quantum mechanical condition that joint distributions are well defined only for commuting observables. I now show that precisely such violations of the restrictive joint distribution structure of quantum mechanics, imposed by the existence of nontrivial commutation relations, are entirely equivalent to the supposition of deterministic hidden variables.

Proposition (1).—Necessary and sufficient for the existence of a deterministic hidden-variables model is the existence of a distribution $P(ABB')$ for the triple A, B, B' and a distribution $P(A'BB')$ for the triple A', B, B' whose marginals yield the experimental probabilities and which, in addition, also yield one and the same joint distribution $P(BB')$ for the noncommuting pair B, B'.

I have already shown necessity above. To show sufficiency I will simply show how to build a distribution $P(AA'BB')$ from the triples $P(ABB')$, $P(A'BB')$, and the common joint $P(BB')$; one that returns them again as marginals. The canonical construction, then, produces the required deterministic hidden-variables model. If we set $P(AA'BB') = [P(ABB')P(A'BB')]/[P(BB')]$, then it is straightforward to verify that this is a proper joint distribution, with the required marginals. [If $P(BB') = 0$, set $P(AA'BB') = 0$.]

The joint distributions for noncommuting observables, that are equivalent to the existence of deterministic hidden variables [according to Proposition (1)], can now be put to work to derive restrictions on the probabilities of a correlation experiment. Using the marginals we have

$$P(ABB') = P(AA'BB') + P(A\bar{A}'BB') \le P(A'B) + P(\bar{A}'B') = P(A'B) + P(B') - P(A'B'), \quad (4)$$

$$P(\bar{A}BB') = P(\bar{A}A'BB') + P(\bar{A}\bar{A}'BB') \le P(A'B') + P(\bar{A}'B) = P(A'B') + P(B) - P(A'B). \quad (5)$$

Then, by straightforward calculation,

$$0 \le P(A\bar{B}\bar{B}') = P(A) - P(AB) - P(AB') + P(ABB'), \quad (6)$$

$$0 \le P(\bar{A}\bar{B}\bar{B}') = 1 - P(A) - P(B) - P(B') + P(AB) + P(AB') + P(\bar{A}BB'). \quad (7)$$

Using the inequality (4) for $P(ABB')$ in (6), and (5) for $P(\bar{A}BB')$ in (7) yields

$$-1 \le P(AB) + P(AB') + P(A'B') - P(A'B) - P(A) - P(B') \le 0. \quad (8)$$

Similar calculations for the other terms in the distributions for A, B, B' and for A', B, B' turn out three more pairs of inequalities. These can be obtained from (8) by first interchanging A with A', then B with B', and finally both A with A' and B with B' together. I shall refer to all eight inequalities thus obtained, collectively, as the Bell/CH inequalities. I have just shown that every deterministic hidden-variables model restricts the probabilites of the experiment so as to satisfy these eight inequalities.

This is well known.[4] Surprisingly, the converse is also true.

Proposition (2).—Necessary and also sufficient for the existence of a deterministic hidden-variables model is that the Bell/CH inequalities hold for the probabilities of the experiment.[5] To establish this it remains to show how to build a deterministic model, given the Bell inequalities. With use of Proposition (1), it will be sufficient to define distributions $P(ABB')$ and $P(A'BB')$ that yield the same distribution $P(BB')$, along with the probabilities of the experiment. For this purpose let β be the minimum of the following (nonnegative) terms: $P(B)$, $P(B')$, $P(AB)+P(B')-P(AB')$, $P(AB')+P(B)-P(AB)$, $P(A'B)+P(B')-P(A'B')$, and $P(A'B')+P(B)-P(A'B)$. Then we will set $P(BB')=\beta$ and define the rest of the distribution for B, B' by $P(B\overline{B}')=P(B)-\beta$, $P(\overline{B}B')=P(B')-\beta$, and $P(\overline{B}\,\overline{B}')=1-P(B)-P(B')+\beta$. One can check that this is well defined, by the choice of β. Then let α be the minimum of β, $\beta-[P(A)+P(B)+P(B')-P(AB')-P(AB)-1]$, $P(AB)$, and $P(AB')$. Similarly, let α' be the minimum of β, $\beta-[P(A')+P(B)+P(B')-P(A'B')-P(A'B)-1]$, $P(A'B)$, and $P(A'B')$. The Bell/CH inequalities guarantee that α and α' are nonnegative. Then we have that $0\leq\alpha\leq\beta\leq 1$ and $0\leq\alpha'\leq\beta\leq 1$. We now set $P(ABB')=\alpha$ and fill out the remainder of the distribution for A, B, B' as follows: $P(AB\overline{B}')=P(AB)-\alpha$, $P(A\overline{B}B')=P(AB')-\alpha$, $P(A\overline{B}\,\overline{B}')=P(A)-P(AB)-P(AB')+\alpha$, $P(\overline{A}BB')=\beta-\alpha$, $P(\overline{A}B\overline{B}')=P(B)-P(AB)-(\beta-\alpha)$, $P(\overline{A}\overline{B}B')=P(B')-P(AB')-(\beta-\alpha)$, and $P(\overline{A}\,\overline{B}\overline{B}')=1-P(A)-P(B)-P(B')+P(AB)+P(AB')+(\beta-\alpha)$. In just the same way we set $P(A'BB')=\alpha'$ and use the same prescription (replacing A by A', and α by α') to fill out the distribution for A', B, B'. Then we must verify that each right-hand term so defined is nonnegative, and no larger than the already defined joint probability for any two of the three observables that occur to the left-hand side of it. That all these inequalities hold follows from the definitions of α, α', and β, and from the assumption that the Bell/CH inequalities hold, together with the usual probabilistic connections governing the experimental probabilities {e.g., that $P(AB)\leq\min[P(A),P(B)]$ or that $P(A)+P(B)\leq 1+P(AB)$, etc.}.[6] One can easily check that $P(ABB')+P(\overline{A}BB')=P(A'BB')+P(\overline{A}'BB')=\beta$, and that the other terms in the distribution of B, B' come out the same whether calculated from the triple A, B, B' or from the triple A', B, B'. Similarly, the experimental distributions for the observables and compatible pairs also come out correctly. Thus the conditions of Proposition (1) for a deterministic hidden-variables model are satisfied if the Bell/CH inequalities hold.

I move, finally, to consider *stochastic hidden-variables models*. Here we relax the requirement that each hidden variable λ determines a unique measurement response (either $+1$ or -1), in favor of the idea that λ only determines the probability of a response. So we replace the response functions of determinism by the probability functions $P(S,\lambda)$ and $P(ST,\lambda)$, where $P(S,\lambda)$ gives the probability that observable S, measured in state λ, yields value $+1$; and $P(ST,\lambda)$ gives that probability that both observables yield $+1$, if measured in λ. [Here S can be replaced by any of A, A', B, and B' in $P(S,\lambda)$ and ST can be replaced by any commuting pair AB, $A'B$, AB', or $A'B'$ in $P(ST,\lambda)$, allowing also for complements.] Then we require that

$$P(S)=\int_\Lambda P(S,\lambda)\rho(\lambda)d\lambda, \tag{9}$$

and

$$P(ST)=\int_\Lambda P(ST,\lambda)\rho(\lambda)d\lambda. \tag{10}$$

If, in addition, the following condition holds

$$P(ST,\lambda)=P(S,\lambda)P(T,\lambda), \tag{11}$$

then the stochastic model is said to be *factorizable*.[7] [Here I use the convention that $P(\overline{S},\lambda)=1-P(S,\lambda)$, and $P(\overline{S}T,\lambda)=P(T,\lambda)-P(ST,\lambda)$, $P(S\overline{T},\lambda)=P(S,\lambda)-P(ST,\lambda)$.]

Clearly, every deterministic hidden-variables model is a factorizable stochastic model, with all probabilities at λ either 0 or 1. Simply set $P(S,\lambda)=1$ if $S(\lambda)=1$ and $P(S,\lambda)=0$ if $S(\lambda)=-1$, and then let $P(ST,\lambda)=P(S,\lambda)P(T,\lambda)$ as in (11). Then trivially, (1) and (2) imply (9) and (10). Although the converse (that every factorizable stochastic model is deterministic) is not true, something close to it is.[8]

Proposition (3).—There exists a factorizable stochastic hidden-variables model for a correlation experiment if and only if there exists a deterministic hidden-variables model for the experiment. One can use a result in the literature to prove this. Clauser and Horne (Ref. 2) show that any factorizable stochastic model satisfies the Bell/CH inequalities, and hence by Proposition (2), it only produces probabilities for a deterministic model. But we can also establish this directly, just by noting that

$$P(AA'BB') = \int_\Lambda P(A,\lambda)P(A',\lambda)P(B,\lambda)P(B',\lambda)\rho(\lambda)d\lambda \tag{12}$$

defines a distribution returning the probabilities of the experiment as marginals, provided (9)–(11) hold.

Proposition (3) shows that, despite appearances, no significant generality is achieved in moving from deterministic hidden variables to stochastic ones, if factorizability is required of the latter: An experiment can be modeled in one way if and only if it can be modeled in the other. Similarly, Proposition (2) shows that, despite appearances, no significant generality is achieved by those derivations of the Bell/CH inequalities that dispense with explicit reference to hidden variables and/or determinism[9]: The assumptions of such derivations imply the existence of deterministic hidden variables for any experiment to which they apply. Finally, I believe that Proposition (1) —conjoined with the other two—shows what hidden variables and the Bell inequalities are all about; namely, imposing requirements to make well defined precisely those probability distributions for noncommuting observables whose rejection is the very essence of quantum mechanics.

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[1]An experiment of this sort is suggested in Sect. 7.2 of the excellent review of the Bell literature by J. F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978). A detailed design of the experiment is given in T. K. Lo and A. Shimony, Phys. Rev. A 23, 3003 (1981). L. S. Bartell, Phys. Rev. D 22, 1352 (1980), shows how to formulate a double-slit experiment in these terms.

[2]J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974). We allow below both for the detection of +1 responses and −1 responses, as in Lo and Shimony (Ref. 1), an allowance to which the Clauser and Horne scheme is readily adapted.

[3]Exactly similar reasoning shows that this conclusion is valid in the more general setting where any number of incompatible observables are measured in R_1 and/or in R_2. Hence the probabilistic assumptions required to show the (in principle) incompatibility of deterministic hidden variables with quantum interference phenomena, in the manner of J. Bub, *The Interpretation of Quantum Mechanics* (Reidel, Dordrecht, 1974), pp. 76–78, are properly available.

[4]Using the fact $P(S)=\frac{1}{2}(\langle S\rangle+1)$ and $P(ST)=\frac{1}{4}(\langle ST\rangle + \langle S\rangle + \langle T\rangle +1)$, we can rewrite (8) as $-2 \leq \langle AB\rangle + \langle AB'\rangle + \langle A'B'\rangle - \langle A'B\rangle \leq 2$. In this form the inequalities were first derived by J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 888 (1969), generalizing a special case discovered by J. S. Bell, Physics (N.Y.) 1, 195 (1964). Since the form of expectation values depends essentially on the particular values of the observables (here ±1), whereas (8) would result whatever the two values, I prefer to work with the purely probabilistic form (8). These inequalities were derived, for the stochastic case, by Clauser and Horne (Ref. 2). The derivation in the text is new.

[5]We can apply Proposition (2) to a hidden-variables model restricted to those particles that enter the collimating apertures of the analyzers. Then, ignoring spurious coincidence counts, the experimental errors are just the errors of detection. Let $0<\epsilon<1$ be the probability of detection for a single particle (assumed the same for each analyzer-detector assembly). If, for perfect efficiency, quantum mechanics predicts that $P(AB)+P(AB')+P(A'B')-P(A'B)=S$ and $P(A)=P(B')=\frac{1}{2}$, then (assuming random errors) inequality (8) would hold for the experimental probabilities just in the case $-1<\epsilon^2 S-\epsilon<0$. In particular for $S>1$ (i.e., a "theoretical" violation of the Bell/CH inequalities) there would be a hidden-variables model, according to Proposition (2), provided $0<\epsilon<1/S$. In the photon correlation experiments (see Clauser and Shimony, Ref. 1, p. 1907 ff) $S=\frac{1}{2}(1+\sqrt{2})=1.207$. Hence for any $\epsilon<0.829$ Proposition (2) yields a deterministic hidden-variables model of the experiment. In the proposed Lo and Shimony molecular experiment (Ref. 1) $S=1.140$; hence any $\epsilon<0.877$ would be compatible with hidden variables. (This bound slightly improves the 0.9 of these authors.) No experiments to date have detector efficiencies exceeding these bounds. Rather, in the cited references a variety of experimental inefficiencies are used to correct the theoretical predictions which, thus corrected, are then judged as violations of the Bell/CH inequalities.

[6]To help sift through the inequalities it may be useful to have two typical cases pointed out. In the case, for instance, $\alpha=\beta=P(A'B')+P(B')-P(A'B)$, then $P(A\bar{B}\bar{B}')>0$ reduces to the right-hand side of (8). In the case $\beta=P(A'B')+P(B)-P(A'B)$ and $\alpha\neq\beta$, $P(AB)$, and $P(AB')$; then $\alpha>0$ reduces to the left-hand side of (8).

[7]I follow the terminology of A. Garuccio and F. Selleri, Lett. Nuovo Cimento 23, 555 (1978). The condition (11), of stochastic independence at each λ, is more often called "locality." But since it is not yet settled whether (11) is, actually, either necessary or sufficient for local causality (in the sense of no superluminary causal signals) I prefer not to pre-judge the issue by using a persuasive terminology. See F. Selleri and G. Tarozzi, Lett. Nuovo Cimento 29, 553 (1980). Geoffrey Hellman, "Stochastic Einstein-Locality and the Bell Theorems" (to be published), also gives a sensitive discussion of the controversial issues. See also the exchange between A. Fine, P. Suppes, and A. Shimony in "PSA: 1980," edited by R. Giere (Philosophy of Science Association, to be published), Vol. 2.

[8]Garuccio and Selleri, Ref. 4, announce the equivalence of deterministic and factorizable stochastic models. But that turns out to be misleading, for they only show that a certain class of linear inequalities holds in the one case if and only if it holds in the other. Proposition (3) below is different, although it implies the

Garuccio and Selleri result. For the special case of approximations via finite frequencies, the equivalence of Proposition (3) is worked out by H. P. Stapp, Epist. Lett. 36, 55 (1978). (My thanks to A. Shimony for calling my attention to this reference.)

[9]H. P. Stapp, Phys. Rev. D 3, 1303 (1971), and P. Eberhard, Nuovo Cimento B 38, 75 (1977), purport to dispense with hidden variables. B. D'Espagnat, Phys. Rev. D 18, 349 (1978), claims to do without determinism.

Calculation of Newton's Gravitational Constant in Infrared-Stable Yang-Mills Theories

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Newton's gravitational constant G is calculated in a class of scale-invariant gauge theories with an infrared fixed point. The sign of G depends on the coefficients in the renormalization-group β function.

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Is Newton's gravitational constant G a fundamental parameter or is it calculable in terms of other fundamental parameters? In this paper I would like to argue the latter view and to present a calculation of G, unfortunately not in the real world, but in a toy world, just to demonstrate that G is indeed calculable.

The form of the non-Abelian gauge field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ dictates that the gauge potential A_μ must have dimension one regardless of the dimension of space-time, and so the Yang-Mills action F^2 must always have dimension four. (It is tempting to suggest that this fact may be connected to the actually observed dimension of space-time.) In contrast, the Einstein-Hilbert action R, being just the scalar curvature, always has dimension two. In this sense, Yang-Mills theory is matched perfectly to the observed four-dimensional space-time while gravity is not. More precisely, if we demand the fundamental theory of the world to be scale invariant, Einstein's theory is excluded. (Furthermore, in a gauge-invariant theory without any fundamental scalar fields, all terms proportional to R such as $\varphi^2 R$ are also excluded.)

It is extremely attractive to impose scale invariance since in a scale-invariant theory with n dimensionless couplings all dimensionless ratios of dimensional physical parameters are calculable[1] in terms of $n-1$ dimensionless couplings. (Some physicists harbor the ultimate ambition that n will eventually be reduced to 1.) Newton's gravitational constant G would then be calculable in terms of a purely flat-space quantity determined by the other interactions. In this context, a formula for G was derived independently by Adler[2] and Zee[3] and reads

$$(16\pi G_{\mathrm{ind}})^{-1} = (i/96)\int d^4x\, x^2 \psi(-x^2) \tag{1}$$

(the subscript "ind" denotes "induced") with $\psi(-x^2) \equiv \langle T\, T(x) T(0)\rangle_0 - \langle T\rangle_0^{\,2}$. This formula expresses G_{ind} in terms of a space-time integral over the vacuum value of the time-ordered product of the trace of the stress-energy tensor $T(x)$.

The philosophy and the physics behind the derivation of Eq. (1) have been amply discussed in the literature[3-7] and will not be elaborated here. If this philosophy is correct, we would be in the exciting position of being able to understand the sign of the gravitational constant.[6] The magnitude of G is merely set by the scale of dynamical scale-invariance breaking.

Actually, the formula in Eq. (1) holds only when the metric is not itself quantized; otherwise, there are extra terms due to fluctuations in the metric which have been worked out by Adler.[5] With the metric quantized, the scale-invariant fundamental action of gravity would consist of a linear combination of R^2, $R_{\mu\nu}^{\;2}$, and $R_{\mu\nu\lambda\rho}^{\;2}$.

In this paper, I treat, for simplicity's sake, the background metric as classical and content myself with studying the formula in Eq. (1). I must mention that this formula is defined only with the understanding that it is to be evaluated with the aid of dimensional regularization. By dimensional considerations, one can see that the expression in Eq. (1) has a quadratic short-distance divergence which is prescribed to be zero by dimensional regularization. After performing a Wick rotation to Euclidean space we write Eq.