Is There a Fundamental Bound on the Rate at Which Information Can Be Processed?

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(Received 29 September 1981)

It is shown that the laws of physics impose no fundamental bound on the rate at which information can be processed. Recent claims that quantum effects impose such bounds are discussed and shown to be erroneous.

PACS numbers: 89.70.+e, 03.65.Bz, 05.90.+m

The speed of electronic information-processing devices being manufactured and designed is continually increasing. It is of interest to inquire whether the rate at which information can be processed is in principle limited only by our ingenuity and by the availability of resources, or whether there exists any fundamental bound imposed by the laws of physics. My purpose here is to explain why there can be no such fundamental bound, and to demonstrate that recent claims to have discovered such bounds are erroneous.

The essential feature of an information-processing device is that packages of information enter it in one state and leave it in another. We may assume that both the initial and the final state are stationary eigenstates of the same commuting set of observables $\hat{O}_i$, in whose values the information resides.

It is sometimes claimed[1] that this assumption is inconsistent with the so-called "time-energy uncertainty principle" $\delta t \delta E \geq \hbar$, interpreted to imply that δt is the least possible time in which a system can be prepared in a state whose energy has variance δE^2; thus δt would be infinite for energy eigenstates. However, it has often been shown by explicit examples [e.g., by Unruh[2]] that this interpretation is simply false and that no such minimum time exists.

In a recent paper,[3] Bekenstein showed that under certain plausible assumptions the entropy-to-energy ratio for spatially bounded systems always satisfies the inequality

$$S/\langle \hat{E} \rangle < (\pi A)^{1/2}. \tag{1}$$

S is the entropy, $\langle \hat{E} \rangle$ the expectation value of the total renormalized energy, and A the area of the smallest sphere circumscribing the system. The symbols $\langle\,\rangle$ denote a renormalized quantum expectation value and, where necessary, a statistical ensemble average. The units are such that $\hbar = c = 1$. This result may be of some importance to our understanding of black-hole thermodynamics though recent work of Unruh and Wald[4] strongly suggests the contrary. In any case it cannot possibly have any relevance to the physics of nongravitating systems. In particular, Bekenstein's own attempt[5] to derive from (1) an upper bound on the rate at which information can be processed by an ideal computer is fallacious. I shall now show that there can be no such bound.

The central argument in Bekenstein's derivation of (1) is as follows. An ensemble of quantum systems at fixed energy attains its maximum entropy in thermodynamic equilibrium. It may then be described by the partition function

$$Z(\beta) = \sum_{n=0}^{\infty} \exp(-\beta E_n). \tag{2}$$

E_n is the total renormalized energy of the nth stationary state. The renormalized zero-point (vacuum) energy E_0 is not in general zero and Bekenstein conjectures that it is positive for all reasonable bounded quantum systems. This conjecture (which I shall not contradict) is crucial in what follows.

The mean renormalized energy and canonical entropy of the ensemble (2) are given by the usual formulas[6,7]

$$\langle \hat{E} \rangle = -Z^{-1} \partial Z/\partial\beta, \tag{3}$$

$$S = \ln Z + \beta \langle \hat{E} \rangle. \tag{4}$$

Thus

$$\frac{S}{\langle \hat{E} \rangle} = \beta - \frac{Z \ln Z}{(\partial Z/\partial \beta)}. \tag{5}$$

Near absolute zero ($\beta \to \infty$), $\langle \hat{E} \rangle$ approaches the positive value E_0, and the third law of thermodynamics states that $S \to 0$, and so the entropy/energy ratio (5) must also approach zero. On the other hand, from (5)

$$\frac{\partial}{\partial\beta}\left(\frac{S}{\langle \hat{E} \rangle}\right) = \frac{Z^2 \ln Z}{(\partial Z/\partial\beta)^2} \langle (\hat{E} - \langle \hat{E} \rangle)^2 \rangle. \tag{6}$$

It is clear from (2) than $\ln Z$ is nonnegative for all sufficiently small values of β. Therefore the entropy/energy ratio, which is nonnegative definite, increases with β for small β. It follows by continuity that the ratio has an upper bound, depending only on the physical contraints on the system.

Bekenstein goes on to demonstrate, by means of an ingenious argument (which I shall not require here), that the relevant constraint is the system's size and that (1), or something very like it, is true of all bounded systems. He expresses surprise that such an apparently far-reaching universal bound has previously gone unnoticed in nonrelativistic thermodynamics. The reason, unfortunately, is that *(1) is inapplicable as it stands to nongravitating systems,* since it refers to the absolute value $\langle\hat{E}\rangle$ of energy, which cannot be observed without gravity. Although (1) does not contain the gravitational constant, the fact that it is not invariant under changes ($\hat{E}\to\hat{E}+\text{const}$) in the zero point of energy reveals its true identity as a purely general relativistic relation. In nongravitational physics, the energy $\hat{E}$ enters observable quantities only through energy *differences.*

In thermodynamic equilibrium, for example, $\hat{E}-\hat{E}_0$ is an important observable, namely the total thermal energy. I am therefore led to investigate the properties of the ratio

$$r(\beta)=S(\beta)/\langle\hat{E}-\hat{E}_0\rangle_\beta, \tag{7}$$

which is meaningful even in the absence of gravity. In analogy to (5) and (6) we have

$$r(\beta)=\beta-\frac{Z\exp(\beta E_0)\ln[Z\exp(\beta E_0)]}{\partial[Z\exp(\beta E_0)]/\partial\beta} \tag{8}$$

and

$$\frac{\partial r(\beta)}{\partial\beta}=\frac{Z^2\exp(2\beta E_0)\ln[Z\exp(\beta E_0)]}{\{\partial[Z\exp(\beta E_0)]/\partial\beta\}^2}\langle(\hat{E}-\langle\hat{E}\rangle)^2\rangle. \tag{9}$$

For large values of β, (2) implies that

$$\begin{aligned}
&Z\exp(\beta E_0)=1+\exp[-\beta(E_1-E_0)]+O(\exp[-\beta(E_2-E_0)]),\\
&\partial[Z\exp(\beta E_0)]/\partial\beta=-(E_1-E_0)\exp[-\beta(E_1-E_0)]+O(\exp[-\beta(E_2-E_0)]),\\
&\partial^2[Z\exp(\beta E_0)]/\partial\beta^2=(E_1-E_0)^2\exp[-\beta(E_1-E_0)]+O(\exp[-\beta(E_2-E_0)]),
\end{aligned} \tag{10}$$

and hence that $\partial r(\beta)/\partial\beta$ approaches unity for sufficiently large β (this conclusion is unaffected if the energy levels are degenerate). Thus, whereas fundamental principles indeed require $S/\langle\hat{E}\rangle$ to be bounded they actually *forbid* the more accessible quantity $r(\beta)$ from so being. [It is claimed in Ref. 3 that the ground (vacuum) state of the packet cannot be used to send information. *If this were true*, the effective $r(\beta)$ (i.e., omitting the ground state from the ensemble average) would have to be bounded just like $S/\langle\hat{E}\rangle$. However, it is not true.]

I now turn to the alleged bound imposed by (1) on the speed of information processors. We shall see that it is $r(\beta)$, not $S/\langle\hat{E}\rangle$, which is relevant. Bekenstein applies the inequality (1) to a "package" of information of mean energy $\langle\hat{E}\rangle$ in transit through a processor. The amount of information I (in bits) in the package cannot exceed $S_{\max}/\ln 2$ where $S_{\max}$ is an upper bound on the entropy of the package. Bekenstein shows by geometrical reasoning that the information rate $\dot{I}$ satisfies

$$\dot{I}<I/\tau<\pi\langle\hat{E}\rangle/\ln 2, \tag{11}$$

where τ is the transit time for the package, which is a lower bound on the time required to process information I. Thus the higher is the required rate of information processing, the larger must be the energy of each information package. But if (Bekenstein goes on to say) "as seems probable, the energy accompanying each 'message' cannot be recycled, the power dissipated ... gives rise to ... an 'overheating' problem." By considering the dependence of various cooling mechanisms on the size, shape, and constitution of the overheated region, he eventually arrives at an approximate bound of 10^{17} s^{-1} on $\dot{I}$, which corresponds to about 10^{15} elementary operations per second for a computer dealing with nine-digit decimal numbers.

The key error in this analysis is the assumption that the zero-point energy of the information package must be dissipated in the processor. In fact, of course, it cannot be dissipated since it is invisible to all but gravitational forces. The true rate at which the processor is heated by Bekenstein's mechanism cannot possibly exceed $(\hat{E}-E_0)/\tau$, a quantity which, it follows from (10), can be made arbitrarily small for fixed $\dot{I}$ by reducing the temperature (increasing β) sufficiently.

Moreover, the assumption, made in both Refs. 1 and 3, that the energy in the incoming information packet is necessarily dissipated in the processor is itself quite false. It is contrary to the often reestablished principle[3, 8-10] that a single observable or commuting set of observables $\hat{O}_i$ may be measured with arbitrarily great precision—and therefore if the packet is initially in an eigenstate of $\hat{O}_i$ no uncontrollable disturbance, and therefore no entropy, need be created by the measurement of $\hat{O}_i$. Thus an arbitrarily small proportion of the packet's energy need in principle be dissipated.

In practice the speed of a processor is of course limited by the availability of desired couplings. But barring quantum gravitational effects (which may set in when the information packet is so energetic that it approaches its Schwarzschild radius, and perhaps give rise to a bound $\dot{I} \lesssim 10^{42}$ s^{-1}), I conclude that there is no fundamental limit on the speed of information processors.

This work was supported in part by the National Science Foundation under Grant No. PHY 7826592.

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