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Drift-Wave Spectra Obtained from the Theory of Nonlinear Ion-Landau Damping in Sheared Magnetic Fields

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The local character, in poloidal mode-number space, of the resonant nonlinear interaction of drift waves with ions in a sheared magnetic field permits an analytical determination of the spectrum. Important processes underlying the stabilization are energy cascade and transfer, respectively, in "close" ($k_\theta' \cong k_\theta$) and "distant" ($k_\theta' a_s \cong 1/k_\theta a_s$) interactions. The spectral index $n=4$ of the high-mode-number tail is independent of the excitation mechanism.

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The nature of the mechanism by which drift waves saturate has been a subject of much controversy in recent fusion research.[1-6] Already in 1969 Sagdeev and Galeev[7] considered the nonlinear scattering off ions (nonlinear ion-Landau "damping") in shearless plasmas. Later Dupree and Tetreault[5] and Krommes[6] presented a renormalized version of ion Compton scattering. We develop the idea further for more realistic fusion environments with magnetic shear. The approach is that of weak turbulence theory, but we retain the scattering contribution of the shielding cloud. There are essential differences from previous works. First, the random-phase approximation is now inadequate in the radial direction since it postulates the statistical independence of the Fourier components. Second, in contrast to the shearless case for which $\partial\omega/\partial k_r \neq 0$, the condition for nonlinear scattering to be efficient, namely that the beat frequency of two natural modes is small in comparison to the individual frequencies, implies for the poloidal mode numbers of the interacting waves either $k_\theta' = k_\theta$ or $k_\theta' a_s = 1/k_\theta a_s$ {in the limit $T_i/T_e \ll 1$, the drift-wave frequency is $\omega = -[k_\theta a_s/(1+k_\theta^2 a_s^2)] \times c_s/L_n$; L_n is the length scale of the density gradient, $c_s = (T_e/m_i)^{1/2}$ is the sound speed, and $a_s = c_s/\Omega_i\}$. We find in agreement with former calculations[6,7] that "close interaction" ($k_\theta' = k_\theta$) yields an energy cascade towards longer wavelengths if $k_\theta a_s < 1$; if $k_\theta a_s > 1$, the cascade process is towards shorter wavelengths. The different behavior in these two spectral regions is related to the group velocity changing sign for $k_\theta a_s = 1$; clearly the group velocity is an important parameter since the scattering concerns only nearby frequencies. The important new result is, however, that "distant interaction" transfers energy from the long ($k_\theta a_s < 1$) to the correspondingly short ($k_\theta' a_s > 1$) wavelengths which permits a stationary solution of the wave equation. It is interesting to note that this process—thus far overlooked—naturally plays the role formerly assigned by Sagdeev and Galeev to wave decay.[7]

The wave kinetic equation is derived by the standard methods of weak turbulence except that the random-phase average in the radial mode numbers is replaced by a selection rule based on the fast radial variation of the linear eigenfunctions: These are assumed to have the Pearlstein-Berk eikonal structure[8] and the method of stationary phase in applied. The result is[9]

$$\frac{T_i}{T_e}\,\mathrm{Im}\,\beta_{\vec{k}} + \gamma_{\vec{k}}\left[\frac{\partial\epsilon(\vec{k},\omega)}{\partial\omega}\right]_{\omega_{\vec{k}}} - \frac{1}{2}\,\frac{1}{|\tilde{n}_{\vec{k}}|^2}\,k_{\parallel}'\,\frac{\partial}{\partial k_r}\left[\frac{\partial\epsilon(\vec{k},\omega)}{\partial k_{\parallel}}\,|\tilde{n}_{\vec{k}}|^2\right]_{\omega_{\vec{k}}} \tag{1}$$

$$= \frac{c^2 T_e^{\,2}}{e^2 B^2}\,\frac{2\pi R}{r|L_s|}\,\mathrm{Im}\sum_{m'}|m'|\int dk_r'\left[\frac{v^2(\vec{k}'',\omega''|\vec{k},\omega_{\vec{k}})}{\epsilon(\vec{k}'',\omega'')} + w(\vec{k},\omega_{\vec{k}}|\vec{k}'',\omega''|\vec{k},\omega_{\vec{k}})\right]|\tilde{n}_{\vec{k}'}|^2\,;$$

L_s is the shear length. $\tilde{n}_{\vec{k}}$ is the Fourier transform of the density fluctuation with the normalization

$$\left[\frac{n(r_0)}{N}\right]^2 = \frac{1}{\Delta r}\sum_{l,m}\int_{r_0-\Delta r/2}^{r_0+\Delta r/2} dr\,|\tilde{n}_l(r - r_{lm})|^2 = \frac{2\pi R}{r|L_s|}\sum_m |m|\int dk_r\,|\tilde{n}_{\vec{k}}|^2 \tag{2}$$

(r_{lm} is the position of the rational surface of toroidal and poloidal mode numbers l and m; note that the number of rational surfaces in an interval Δr is $|m|\Delta r R/r|L_s|$). $\beta_{\vec{k}}\tilde{n}_{\vec{k}} = (\tilde{n}_{\vec{k}})_{\mathrm{el}}{}^{\mathrm{NA}}$ is the "nonadiabatic" electron response. $\omega'' = \omega_{\vec{k}} - \omega_{\vec{k}'}$, $\vec{k}'' = \vec{k} - \vec{k}'$. The term

$$w \equiv (\vec{k}\times\vec{k}'\cdot\hat{n})^2\int dv\,J_0^2(k)J_0^2(k')F_i^M g_{\vec{k}}\,g_{\vec{k}''}[(\omega_{\vec{k}} - \omega_{i\vec{k}}^*)g_{\vec{k}} - \vec{k} \leftrightarrow \vec{k}'],$$

where $g_{\vec{k}} \equiv (\omega_{\vec{k}} - k_{\parallel}v_{\parallel})^{-1}$ and $\hat{n} = \vec{B}/B$, represents the Compton contribution to the scattering process and agrees with the result of Krommes[6] in the weak turbulence limit. The term v^2/ϵ,

$$v \equiv \vec{k}\times\vec{k}'\cdot\hat{n}\int dv\,J_0(k)J_0(k')J_0(k'')F_i^M g_{\vec{k}''}[(\omega - \omega_{i\vec{k}}^*)g_{\vec{k}} - k \leftrightarrow -k'],$$

represents the shielding contribution. [We have used the relation $v(\vec{k}'',\omega''|\vec{k},\omega_{\vec{k}}) = -v(\vec{k},\omega_{\vec{k}}|\vec{k}'',\omega'')$ in writing Eq. (1).] Finally the dielectric function (with adiabatic electrons) is

$$\epsilon(\vec{k},\omega) \equiv 1 + (T_i/T_e) - \int dv\,J_0^2(k)F_i^M(\omega - \omega_{i\vec{k}}^*)g_{\vec{k}}\,.$$

[We do not consider the effect of the temperature gradient and $J_0(k) \equiv J_0(k_\perp v_\perp/\Omega_i)$.]

To proceed, we first take profit of the local character of the interaction to expand $k_\theta' = \langle k_\theta'\rangle + \delta k_\theta'$ and any function thereof where $\delta k_\theta' \equiv \delta k_\theta'(\omega'')$ and $\langle k_\theta'\rangle$ = either k_θ (close interaction) or $1/k_\theta \times a_s^2$ (distant interaction). We then introduce the hypothesis that the radial spectrum can be taken over from linear theory to eliminate the integral character of Eq. (1) in k_r. Finally we evaluate integrals of the form $\int d\omega''\,\omega''\epsilon^{\pm 1}(\langle\vec{k}''\rangle,\omega'')$ which are left over by a residue method.[10] We have been careful to exclude the contribution from the poles of $\epsilon^{-1}(k'',\omega'')$ on the real axis which correspond to absorption in *three-wave* scattering processes discussed elsewhere.[3,9,11,12] In this way Eq. (1) is reduced to the differential form

$$L_1[f(y)] + L_2[f(y^{-1})] = s(y)\,, \tag{3}$$

where the function $f(y)$ is related to the one-dimensional poloidal spectrum of the density fluctuation via $|\tilde{n}|^2 = \int dk_\theta |\tilde{n}|_{k_\theta}^{\,2}$,

$$|\tilde{n}|_{k_\theta}^{\,2} = \frac{2\pi R r}{|L_s|}\,y^{-1} f\,. \tag{4}$$

The variable $y \equiv k_\theta^2 a_s^2$; the operators $L_1 \equiv y\,\partial/\partial y + (2-y)/(1-y)$ and $L_2 \equiv (2-y)^2\,\partial/\partial y + (2-y)/(1-y)$. A singularity appears at $y = 1$ because the transformation $\delta k_\theta' = \delta k_\theta'(\omega'')$ and the Jacobian $d\,\delta k_\theta'/d\omega''$ have been approximated by linear forms which procedure is invalid in the vicinity of $y = 1$ (where $\partial\omega_{\langle\vec{k}'\rangle}/\partial\langle k_\theta'\rangle$ vanishes). The singularity reflects in an approximate way the fact that the number of quantum energy levels involved in the interaction becomes infinite there and is by no means related to a failure of the weak-turbulence expansion. The source term is

$$s(y) \equiv -\frac{\sigma}{\Delta\langle x^2\rangle}(\tilde{\gamma}_{\vec{k},L} - \tilde{\gamma}_{\vec{k}})\frac{(1-y)^2}{(1+y)^3}y^{-3/2}, \tag{5}$$

where $\Delta \equiv 24\pi^2\tau^2 \gamma R|L_n|/a_s^3$, $\tau \equiv T_i/T_e$, $\sigma = \mathrm{sign}(1-y)$, and L_n is the density gradient length scale. $\langle x^2\rangle$ is related to the fourth power of the cutoff radial mode number set by linear ion-Landau damping; we estimate $\langle x^2\rangle\tau^2 \simeq 1/250$. The linear growth rate is given by $\gamma_{\vec{k}}(\text{linear}) = \tilde{\gamma}_{\vec{k},L}|\omega_{\vec{k}}||L_n/L_s|$. In Eq. (5) $\gamma_{\vec{k}}$ represents the total damping rate (linear and nonlinear) in those regions of k_θ space where the turbulence is not excited; where this is the case, the noise level is proportional to the Cherenkov and bremsstrahlung radiation and inversely proportional to $\gamma_{\vec{k}}$.[12]

The general solution of Eq. (3) is

$$f(y) \equiv (\sigma/\Delta\langle x^2\rangle)[(a_1 - \tfrac{1}{8}i_1)f_1 + (a_2 + \tfrac{1}{5}i_2)], \tag{6}$$

where a_1 and a_2 are integration constants,

$$f_1 \equiv -\frac{2}{5}\frac{y-4}{R^{1/2}}, \quad R \equiv -y^2 + 3y - 1, \tag{7}$$

$$i_1(y) \equiv i_1(y^{-1}) = \int_1^y \frac{dy'}{[R(y')]^{1/2}}\frac{y'}{(1+y')^3}[y'^{-5/2}(4y'-1)\tilde{\theta}(y') - y'^{5/2}(4/y'-1)\tilde{\theta}(y'^{-1})] \tag{8}$$

and

$$i_2(y) \equiv i_2(y^{-1}) = \int_1^y dy'\frac{(y')^{1/2}}{(1+y')^3}[y'^{-2}\tilde{\theta}(y') - y'^2\tilde{\theta}(y'^{-1})]. \tag{9}$$

We have defined $\tilde{\theta}(y) \equiv \tilde{\gamma}_L(y) - \tilde{\gamma}(y)$. The solution is subject to the constraints $\gamma_{\vec{k}} \equiv 0$ wherever $f > 0$ and $\gamma_{\vec{k}} < 0$ wherever $f \equiv 0$. The additional freedom introduced by the retention of the damping rate $\gamma_{\vec{k}}$ enables us to exclude unphysical negative solutions. To avoid any discontinuous behavior we require $f(1) = 0$; to exclude unphysical (homogeneous) solutions we impose $|\partial f/\partial y|_1 = 0$. Physically this choice of boundary conditions results from the reversal of the energy flow at $y = 1$. To show that the roots y_1 and $y_2 = y_1^{-1}$ of the polynomial R are not singular points of the solution suppose first that $f(y) > 0$ both to the right of y_1 and to the left of y_2. $\tilde{\gamma}(y)$ should then be identically zero in both neighborhoods. But since $\lim(y \to y_{1,2}) f_1 = \infty$, $f(y)$ will take on opposite signs in these neighborhoods (because of the sign factor σ) which contradicts the premise. Hence $f \equiv 0$ and $\tilde{\gamma} < 0$ either around y_1 or around y_2. $f \equiv 0$ in a given range $a < y < b$ is the requested (integral) equation for the damping rate in this range. The solution is

$$\tilde{\gamma}(y) = -2y^{3/2}(1+y)^3(y-2)[\tfrac{1}{2}\alpha_0 + h_0(y^{-1})] + \tilde{\gamma}_L(y) - y^3(y-2)^2\tilde{\gamma}_L(y^{-1}), \tag{10}$$

where

$$h_0(y) \equiv \int_1^y \frac{dy'}{(y')^{1/2}}\frac{(y'-1)}{(1+y')^3}\tilde{\gamma}_L(y'). \tag{11}$$

In the complementary domain $a^{-1} > y > b^{-1}$, the function f reduces to the simple form

$$f \equiv (\sigma/\Delta\langle x^2\rangle)[\tfrac{1}{2}\alpha_0 + h_0(y)](1-y)/y^2, \tag{12}$$

where α_0 is a matching constant. Equations (10) and (12) demonstrate the regularity of the solution at the roots y_1, y_2 of $R = 0$.

To illustrate the theory we have calculated the spectrum, the turbulence level, and the anomalous transport for a concrete excitation model. We assume

$$\tilde{\gamma}_L(y) = 2Dy/(1+y)^2 - 1 \tag{13}$$

which fairly well describes the universal instability: The substracted part represents the effect of shear; the numerical factor $2D \equiv \alpha|L_s/L_n|^{3/2} \times (m_e/m_i)^{1/2}\ln(\lambda/X_e)$ where the logarithmic term $\simeq 2.5$.[13] [The (non)existence of this instability is a complicated story. Personally we lean to a view broadly similar to that adopted by Hirshman and Molvig.[4] In the worst case (13) will still be a proper mathematical model. We consider α as a fitting parameter.]

The solution corresponding to (13) where $D = 5$ is displayed in Fig. 1; for comparison we have also shown the result obtained when distant interaction is not taken into account. The following conclusions emerge:

(1) The region of suprathermal fluctuations do not coincide with those of linear instability.

(2) The energy source or sink available at high

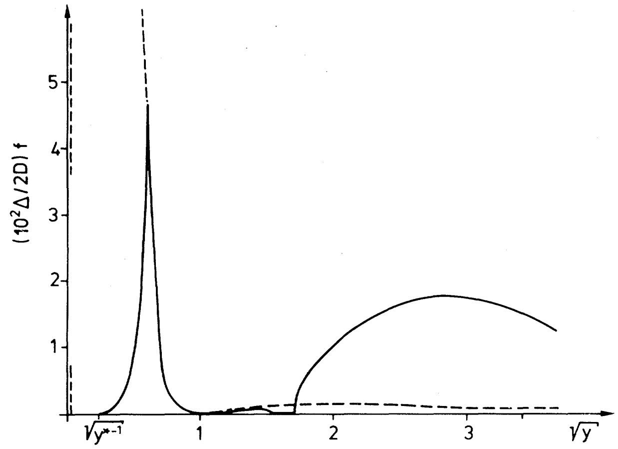

FIG. 1. Electric field energy density $|\tilde{e}|_{k_\theta}^2 \sim f$ vs normalized mode number $y^{1/2} \equiv k_\theta a_s$. Dotted line is the result obtained with close interaction only. Continuous line is with both close and distant interactions taken into account.

mode numbers has very little influence upon the spectrum. This statement can be inferred by comparing the weight factors of the contributions from $\tilde{\gamma}_L(y)$ and $\tilde{\gamma}_L(y^{-1})$ in Eqs. (8) and (9). It has been verified numerically using (13) for $y < 1$, but $\tilde{\gamma}_L(y) \equiv 0$ for $y > 1$.

(3) The poloidal density spectrum, Eq. (4), behaves asymptotically as k_θ^{-4} (for $y > y^* > 1$, see Fig. 1). This behavior, which stems from the integral $h_0(y)$ approaching a constant at large y's, is universal, i.e., independent of the instability model, and in good agreement with results obtained on the Microtor tokamak,[14] where a spectral index 3.5 was measured.

(4) While 75% of the squared density fluctuation arises from the long-wavelength modes ($k_\theta a_s < 1$), the high mode numbers contribute for 85% to the diffusion. This result is interesting since the experimental level of fluctuations measured in the tokamak at Fontenay-aux-Roses[15] and in some other machines at the long-wavelength end of the spectrum (the only one often accessible by microwave probing) falls short, by a factor 5–10, of explaining the convective energy losses calculated from the power balance equation.

(5) Rough agreement is obtained both concerning the energy confinement time and the turbulence level for the Princeton Large Torus experiment described in Ref. 16, if the growth model (13) is used with the fitting parameter $\alpha \simeq 2$–4. In contrast *simultaneous* agreement on these parameters cannot be reached if distant interaction is not taken into consideration.

(6) The tubulence level

$$|\tilde{n}|^2 \equiv (12\pi\tau^2\langle x^2\rangle)^{-1}(a_s^2/|L_sL_n|)\int dy\, y^{-3/2}(f\Delta)$$

is smaller than that obtained in Refs. 2 and 4 by the ratio $|L_n|/|L_s|$ (and a numerical factor smaller than unity).

(7) The microscopic expression of the equivalent diffusion coefficient roughly scales as $\chi_e \sim T_e^{3/2}/m_i^{1/2}qRJ^2$ (the shear length $L_s = qR$; $q \equiv$ safety factor; $J \equiv$ current density). Under the assumption of Ohmic heating the scaling $\chi_e \sim J^2/NT_e^{5/2}$ results from power balance. Eliminating J yields $\chi_e \sim 1/(NT_e qRm_i)^{1/2}$ which is not very different from admitted scaling laws. Note also that $T_e \sim J\alpha(qRm_i^{1/2}/N)^{1/4}$ ($\alpha \equiv$ plasma radius).

(8) The present theory cannot account for the observed spectral width in frequency because of the neglect of incoherent nonlinear wave emission.[3,9,11,12]

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