Magnetic Spectra and Electron Transport of Current-Carrying Plasmas

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A statistical theory for the properties of current-carrying plasma is postulated. The small-scale magnetic turbulences are isotropic in $\vec{k}$ and have a wave-number spectrum similar to that of black-body radiation. The electron transport coefficient is found to have an n^{-1} dependence and a numerical value remarkably close to that of the empirical Alcator scaling. The frequency spectrum (as seen by a magnetic probe) is also calculated and is in good agreement with experimental results. Some unresolved questions are discussed.

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Current-carrying plasma has been used for magnetic fusion research in several approaches, e.g., tokamaks and different types of pinches. In these devices, gases are ionized and heated by discharge currents to form high-temperature plasmas. In most cases the self-fields of plasma currents also play an essential role in confinement. Because of the way plasmas are created, it is conceivable that plasmas probably will be in turbulent states in these devices. Indeed, experimental results accumulated from tokamaks during the past few years indicate that the electron transport follows a persistent anomalous n^{-1} scaling, the Alcator scaling[1]

$$nD_r = (5-80)\times 10^{17}/\text{sec cm}, \tag{1}$$

where n is the plasma density in units of cm^{-3}. At the same time, the importance of magnetic field fluctuations on electron transport has been stressed theoretically.[2-11] Recently, experiments aimed to correlate the magnetic fluctuation and transport have shown that broadband magnetic spectra exist in tokamaks.[12,13]

Here I postulate a statistical theory which gives an n^{-1} scaling and a broadband magnetic spectra. The results are in good agreement with experimental observations.

Evidence and arguments favoring the statistical mechanics treatment of plasma turbulence can be found in an excellent article by Montgomery, Turner, and Vahala.[14] The invariance of magnetic helicity[14-17]

$$H_m = \int \vec{A}\cdot\vec{B}\,dV \tag{2}$$

is the foundation of our calculation. When we drive a dc current in a plasma, the $\omega\approx 0$ collective plasma modes are preferentially excited. These collective modes are able to exchange energy and magnetic helicity with the external circuit. I hypothesize that these $\omega\approx 0$ magnetic fields in plasma interact continuously with the external circuit and approach an equilibrium. This situation is analogous to that of electromagnetic radiation inside a blackbody cavity where photons are continuously absorbed and reemitted by the walls and eventually reach an equilibrium which depends solely on the temperature of the walls of the cavity.

I adopt the Chandrasekhar-Kendall functions[14,18] for the representation of these $\omega\approx 0$ magnetic fields. Namely, I expand the magnetic fields in terms of the solutions of the eigenvalue equation

$$\nabla\times\vec{B}_s = \mu_s\vec{B}_s, \tag{3}$$

with

$$\vec{B}_s\cdot\hat{n} = 0 \tag{4}$$

on the plasma boundary, where $\hat{n}$ is a unit vector normal to the plasma, and $\vec{B}_s$ and μ_s are the eigenfunction and eigenvalue of state s, respectively. It can be shown[19] that $\vec{k}_s\cdot\vec{B}_s=0$ and $k_s=|\vec{k}_s|=|\mu_s|$, where $\vec{k}_s$ is the wave vector of field $\vec{B}_s$. It is also straightforward to show that

$$\int\frac{|\vec{B}_s|^2}{8\pi}dV = \frac{\mu_s}{8\pi}\int\vec{A}_s\cdot\vec{B}_s\,dV = \mu_s\frac{H_s}{8\pi}. \tag{5}$$

Based on this observation, we may visualize the magnetic fields as a collection of magnetic structures, which we will call "magnetors." Each magnetor carries a magnetic helicity h_0 and an energy $\epsilon_s = k_s(h_0/8\pi)$. Here we use the convention $h_0>0$ to ensure that energy >0. By postulating that the magnetors (in equilibrium with the external circuit) obey the Gibbs distribution, we have for the mean number of magnetors in a particular state s

$$\bar{n}_s = \frac{1}{\exp(k_s/k_T)-1}, \tag{6}$$

total magnetic helicity

$$H_m = \int \vec{A}\cdot\vec{B}\, dV = \sum_s \frac{h_0}{\exp(k_s/k_T)-1}, \tag{7}$$

and total magnetic energy

$$E_m = \int \frac{B^2}{8\pi}\, dV = \sum_s \frac{k_s h_0/8\pi}{\exp(k_s/k_T)-1}, \tag{8}$$

where k_T is a constant characterizing this equilibrium. Notice that E_m/H_m is a monotonically increasing function of k_T. Because of the boundary condition $\vec{B}\cdot\hat{n}=0$, there are relatively few allowable long-wavelength (comparable to the plasma dimension) modes, and, in general, they are not isotropically distributed in $\vec{k}$ space.[14,17] On the other hand, small-scale (high-energy) modes which are important for transport are not sensitive to the boundary condition and therefore are distributed isotropically in $\vec{k}$ space. For these short-wavelength modes, the summation over state s can be converted into integration in $k=|\vec{k}|$. Thus

$$\sum_{s'} \frac{k_s h_0/8\pi}{\exp(k_s/k_T)-1} \to \int \frac{(kh_0/8\pi)V}{\exp(k/k_T)-1}\frac{d^3k}{(2\pi)^3} = \frac{Vh_0}{2(2\pi)^3}\int \frac{k^3\,dk}{\exp(k/k_T)-1}, \tag{9}$$

where $\sum_{s'}$ means summation over the short-wavelength modes and V is the confinement volume. This wavelength spectrum is the same as that of blackbody radiation.

Because of their considerably smaller mass, the plasma current is primarily carried by electrons. In the presence of magnetic turbulence, the magnetors, the electron fluid acquires random velocities along the small-scale magnetic field lines,

$$\tilde{\vec{v}}_d = \sum_{s'} \frac{\vec{j}_s}{ne} e^{i\vec{k}_s\cdot\vec{r}} = \sum_{s'} \frac{ck_s}{4\pi ne}\vec{B}_s e^{i\vec{k}_s\cdot\vec{r}}. \tag{10}$$

The diffusion coefficient can be calculated from

$$D = \int_0^\infty \langle \tilde{\vec{v}}_d(t)\cdot\tilde{\vec{v}}_d(t+\tau)\rangle d\tau. \tag{11}$$

By substituting Eq. (10) into Eq. (11) and using the diffusing orbital theory,[20,21] we have

$$D = (c/4\pi ne)\tilde{b}, \tag{12}$$

where $\tilde{b} = (\sum_{s'} B_s^2)^{1/2}$ is the rms total small-scale field. Since the turbulence is isotropic, the radial diffusion is $D_r = \frac{1}{3}D$, and

$$nD_r = 1.66\times10^{18}[\tilde{b}/(1\ \mathrm{G})]\ (\mathrm{cm\ sec})^{-1}. \tag{13}$$

Order-of-magnitude estimates of $\tilde{b}$ range from a tenth of a gauss to several gauss in tokamaks. With these values Eq. (13) gives nD_r's that are not far from the Alcator scaling Eq. (1). However, since the relation between $\tilde{b}$ and other plasma parameters (n, T, B, etc.) has not been established yet, Eq. (13) may not be viewed as a "true" scaling law. The confinement time τ_e can also be calculated from Eq. (13):

$$\tau_e = a^2/D_r = 6.03\times10^{-19} na^2/\tilde{b}\ \mathrm{sec\ (cgs)}. \tag{14}$$

Because of electron transport, an individual mode has a correlation time of $(Dk^2)^{-1}$, i.e., it decays in this time scale. However, since it is in equilibrium with external circuits, it is regenerated on the same time scale to maintain a constant average amplitude. Therefore, to the magnetic probe, it appears to have an effective frequency approximately

$$\omega_k \simeq \tfrac{1}{2}Dk^2. \tag{15}$$

Substituting this frequency into Eq. (9), we transform it into frequency spectrum for the small-scale modes:

$$B^2(\omega)d\omega = \frac{Vh_0 k_T^4}{4\pi^2\omega_T}\left(\frac{(\omega/\omega_T)}{\exp[(\omega/\omega_T)^{1/2}]-1}\right)d\omega, \tag{16}$$

where

$$\omega_T = \tfrac{1}{2}Dk_T^2 = \tfrac{3}{2}k_T^2a^2/\tau_e. \tag{17}$$

Operationally, we can adjust two parameters in Eq. (16), ω_T and amplitude, to get a best fit of experimental data, and then we can determine the corresponding τ_e and $k_T a$ from Eqs. (9), (12), (14), and (17). Here we proceed differently to demonstrate the theory. We notice that the relative fluctuating amplitude $\tilde{b}/B_0$ is a very sensitive function of $k_T a$. From theoretical calculations, $k_T a \sim 0.8$ gives $\tilde{b}/B_0 \sim 10^{-4}$, a typical value for tokamaks, and $k_T a \sim 1.2$ gives $\tilde{b}/B_0 \sim 10^{-3}$ to 10^{-2}, values appropriate for pinches. Therefore, we simply use $k_T a = 0.8$ for tokamaks and $k_T a = 1.2$ for pinches, and we then calculate ω_T from Eq. (17) by using either experimental τ_e, if available, or τ_e from Eq. (14). We first compare our prediction with the Macrotor tokamak measurements[12] of $B_r(\omega)$. $B_r(\omega)$ can be calculated from Eq. (16):

$$B_r(\omega) = \left(\frac{B^2(\omega)}{3}\right)^{1/2} = \left(\frac{Vh_0k_T^4}{12\pi^2\omega_T}\right)^{1/2}\left(\frac{(\omega/\omega_T)}{\exp[(\omega/\omega_T)^{1/2}]-1}\right)^{1/2}. \tag{18}$$

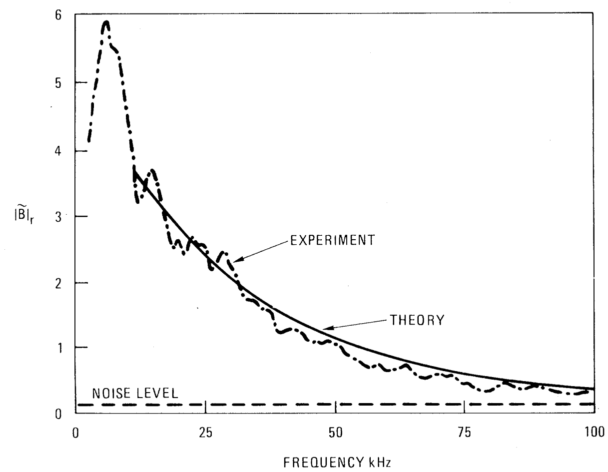

FIG. 1. Comparison of the theoretical $B_r(\omega)$ and the measurement on Macrotor (Ref. 12).

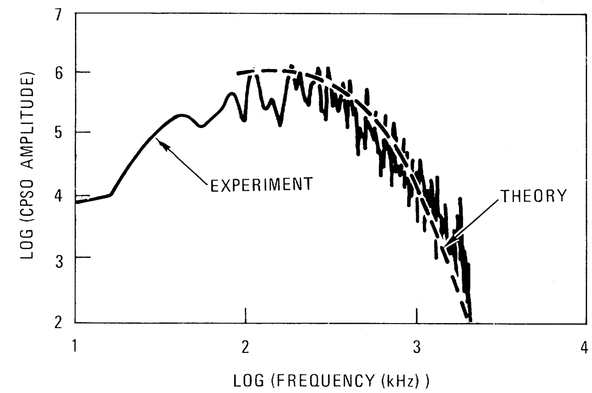

FIG. 2. Comparison of the theoretical $\dot{B}_r^2(\omega) = \omega^2 B_r^2(\omega)$ and the measurement at the California Institute of Technology (Ref. 13).

By using $k_T a = 0.8$ and $\tau_e = 10^{-3}$ sec,[12] we have $\omega_T = 10^3$ Hz. Using this ω_T and adjusting the amplitude of the theoretical curve to the experimental data at 30 kHz, we have the comparison of $B_r(\omega)$ in Fig. 1. We next consider a measurement by Hedemann and Gould.[13] The measurement is at 7 msec in a tokamak discharge with 25-kA plasma current. With $n = 2\times10^{12}/\mathrm{cm}^3$, $a = 15$ cm, and $\tilde{b} = 1$ G, τ_e is estimated from Eq. (14) to be 2.7×10^{-4} sec.[22] Using $k_T a = 0.8$, ω_T is calculated to be 3.5×10^3 Hz. The theoretical result is shown together with the experimental data for $\dot{B}_r^2(\omega) = \omega^2 B_r^2(\omega)$ in Fig. 2 where the theoretical curve is calibrated with the experimental data at 100 kHz. As we can see, the agreement in both cases is unusually good.

A broadband magnetic spectrum has also been observed on the reversed-field pinch ZETA.[23] With $n = 5\times10^{13}/\mathrm{cm}^3$, $a = 45$ cm, $\tilde{b} = 10$ G, the confinement time is estimated from Eq. (14) to be 6.1 msec. By taking $k_T a = 1.2$, ω_T is calculated to be 354 Hz. The comparison of $B^2(\omega)$ is shown in Fig. 3. The calibration of the theoretical curve to the experimental point is made at 15 kHz. The broken line is the theory of Eq. (16) whereas the solid line is the experimental fitting of the data points (dots).

It is amazing that this simple, homogeneous-fluid treatment gives such a good fit to all these experiments. However, some questions remain to be solved. For example, this theory cannot account for the two-dimensional features as seen in experiments,[12,13,23] and kinetic effects have not been studied yet.

In summary, a statistical theory for magnetic turbulence of current-carrying plasmas has been presented. The wavelength spectrum is similar to that of blackbody radiation. By use of diffusing-orbital theory, the electron transport and frequency spectrum of small-scale magnetic fluctuations are calculated and both are in good agreement with experimental results. The electron transport scales like $n^{-1}\tilde{b}$.

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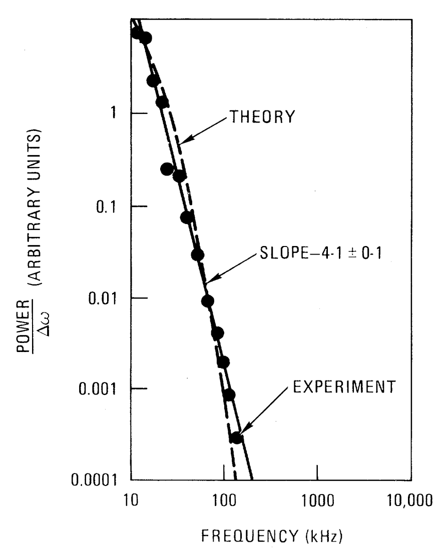

FIG. 3. Comparison of the theoretical $B^2(\omega)$ and the measurement on ZETA (Ref. 23).

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Drift-Wave Spectra Obtained from the Theory of Nonlinear Ion-Landau Damping in Sheared Magnetic Fields

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The local character, in poloidal mode-number space, of the resonant nonlinear interaction of drift waves with ions in a sheared magnetic field permits an analytical determination of the spectrum. Important processes underlying the stabilization are energy cascade and transfer, respectively, in "close" ($k_\theta' \cong k_\theta$) and "distant" ($k_\theta' a_s \cong 1/k_\theta a_s$) interactions. The spectral index $n=4$ of the high-mode-number tail is independent of the excitation mechanism.

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The nature of the mechanism by which drift waves saturate has been a subject of much controversy in recent fusion research.[1-6] Already in 1969 Sagdeev and Galeev[7] considered the nonlinear scattering off ions (nonlinear ion-Landau "damping") in shearless plasmas. Later Dupree and Tetreault[5] and Krommes[6] presented a renormalized version of ion Compton scattering. We develop the idea further for more realistic fusion environments with magnetic shear. The approach is that of weak turbulence theory, but we retain the scattering contribution of the shielding cloud. There are essential differences from previous works. First, the random-phase approximation is now inadequate in the radial direction since it postulates the statistical independence of the Fourier components. Second, in contrast to the shearless case for which $\partial\omega/\partial k_r \neq 0$, the condition for nonlinear scattering to be efficient, namely that the beat frequency of two natural modes is small in comparison to the individual frequencies, implies for the poloidal mode numbers of the interacting waves either $k_\theta' = k_\theta$ or $k_\theta' a_s = 1/k_\theta a_s$ {in the limit $T_i/T_e \ll 1$,

