Nonlinear Theory of Ballooning Modes

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The influence of three-wave interaction on stability of ballooning modes is determined in the presence of magnetic shear and a fully toroidal large-aspect-ratio field geometry. From the ideal two-fluid equations, possibilities for nonlinear instability of the explosive type are established.

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Stability properties of ballooning modes are of great importance in determining the maximum achievable plasma β in a tokamak reactor.[1-5] However, except for some preliminary attempts,[6-7] only linear theories have been reported, leaving unanswered important questions on nonlinear interaction. Recently, such questions have acquired new urgency in view of the prediction[8] of explosively unstable drift-Alfvén waves in a low-β, low-ion-temperature plasma with slab field geometry, without shear. Although ballooning modes belong to the same general class of waves, the toroidal structure of the magnetic field geometry is essential for their existence. Furthermore, the effect of shear in linear theory[4,5,9] prompts an investigation of the influence of shear on nonlinear stability as well. Also, it is of interest to develop a theory capable of covering the second stability region,[5] whereas even for the boundary of the first stability region, finite-β effects have to be taken into account systematically. Finally, with regard to present-day tokamak operations, the assumption of low ion temperature has to be removed.

The above considerations led us to describe three-wave interactions between ballooning modes in high-β tokamak configurations. The ideal two-fluid model is adopted, with finite-Larmor-radius effects being taken into account only in an orbit-averaged sense,[10] under the assumption $\rho_i/L_\perp \ll 1 \gg \omega/\omega_{ci}$, where $L_\perp$ is the perpendicular gradient scale length of the eigenfunction. Since, intrinsically, the curvature and parallel wave vectors are of the same order, curvature effects are incorporated in a systematic manner, without recourse to a pseudogravitation. Only high mode numbers are considered ($n^{1/2} \gg 1$), since these are believed to be characteristic for the worst instabilities.[2-5]

Up to and including first order in the polarization and finite-Larmor-radius drift velocities, the ion equation of motion

$$n_i m_i d\vec{v}_i/dt = -\nabla\cdot\overleftrightarrow{\Pi}_i + en_i\vec{E} + en_i\vec{v}_i\times\vec{B}/c \qquad (1)$$

is solved for the component of the velocity perpendicular to the unperturbed magnetic field, in terms of the parallel velocity and the electromagnetic fields. In the electron equation of motion, polarization and gyroviscous drifts are neglected ($\rho_e/L_\perp \simeq 0 \simeq \omega/\omega_{ce}$), i.e.,

$$0 \simeq -\nabla P_e - en_e\vec{E} - en_e\vec{v}_e\times\vec{B}/c. \qquad (2)$$

In (1) and (2), n_j, m_j, and $\vec{v}_j$, $j=i, e$, are the ion and electron particle number densities, mass, and fluid velocities, respectively; $-e$ is the electron charge, c is the velocity of light *in vacuo;* $\overleftrightarrow{\Pi}_i \equiv \int d^3v\, \vec{v}\vec{v} f_i$ is the ion stress tensor, and P_e is the electron pressure.

The representation of electromagnetic fields, $\vec{E}$ and $\vec{B}$, by potentials is simplified by the assumption of high mode number and low frequency ($|\omega|^2/k_\perp^2 V_A^2 \ll 1$), since then

$$4\pi\delta P + B\delta B_\parallel \simeq 0, \qquad (3)$$

where P is the sum of ion and electron kinetic pressure. Since the parallel gradient operator is invertible for ballooning modes, (3) can be satisfield by the *Ansatz*

$$\delta\vec{A} = -\alpha\nabla_\perp\delta\psi, \qquad (4)$$

with the choice

$$\alpha = \exp(-\tfrac{1}{2}\int_0^r d\rho\,\beta d\ln P/d\rho).$$

Here, terms of order $\epsilon k_\parallel/k_\perp$ were neglected [$\epsilon \sim \beta \sim k_\parallel a \sim 1/(k_\perp a) \sim a/R \sim B_p/B$, a = minor radius, R = major radius, B_p = poloidal field, β = plasma β, $k_{\parallel,\perp}$ = parallel and perpendicular wave numbers]. Also, in evaluating δP, the combined effect on $\delta B_\parallel$ of finite Larmor radius and plasma β, due to a difference between fluid and field displacement, has been neglected.

Within the context of the present field representation, combination of the aforementioned expressions for the perpendicular ion and electron velocities with the induction law yields one equation between the electrostatic potential $\delta\varphi$ and the potential $\delta\psi$ introduced in (4). Under the assumption that finite ion inertia and Larmor radius effects only result in small differences between the ion and electron pressure perturbations, an estimate can be obtained for $\delta\varphi$, from pressure balance and the electron equation of motion. If one assumes that $k_\perp a \gg 1 \gg |\nabla\cdot\vec{e}_\parallel|/k_\parallel$, the dominant part of $\delta\varphi$ is seen to equal

$$\delta\varphi \simeq -\frac{B}{8\pi en}\vec{e}_\parallel\cdot(\nabla\times\alpha\nabla_\perp\delta\psi) + \vec{V}_{de}{}^0\cdot\nabla\delta\psi/c \simeq 0,$$

since $\vec{V}_{de}{}^0 \equiv (c\nabla P_e{}^0/enB^0)\times\vec{e}_\parallel$, with $\vec{e}_\parallel \equiv \vec{B}^0/B^0$. Hence, $\delta\varphi \sim \epsilon\omega\delta\psi/c$ and can be neglected in the expressions for the perpendicular fields. Thereby we arrive at the following closed equation for the potential $\delta\Phi \equiv (\alpha\omega/ic)\delta\psi$ ($\Delta \equiv \nabla^2$, $\Delta_\perp \equiv \nabla_\perp{}^2$):

$$[\partial^2/\partial t^2 + (\partial/\partial t)\vec{V}_{di}{}^0\cdot\nabla - V_A{}^2\nabla_\parallel{}^2]\Delta_\perp\delta\Phi + V_A{}^2\nabla_\parallel(\vec{e}_\parallel\cdot\Delta\nabla_\perp\delta\Phi) - 2\omega_{ci}(\vec{e}_\parallel\times\vec{\kappa})\cdot\nabla(\vec{V}_{de}{}^0\cdot\nabla)\delta\Phi$$
$$= -(\partial/\partial t)(\delta\vec{V}_E + \delta\vec{V}_{di})\cdot\nabla\Delta_\perp\delta\Phi - 4\pi V_A{}^2/c^2(\partial/\partial t)\nabla\cdot(J_\parallel\vec{B}_\perp/B^0). \qquad (5)$$

Here, $\delta\vec{V}_E$ and $\delta\vec{V}_{di}$ are the perturbed cross-field and ion diamagnetic drift velocities, $\vec{\kappa} = \vec{e}_\parallel\cdot\nabla\vec{e}_\parallel$ is the curvature vector, V_A is the unperturbed Alfvén velocity, and ω_{ci} and $\vec{V}_{de}{}^0$ are the unperturbed ion cyclotron frequency and electron diamagnetic drift velocity, respectively. On the left-hand side of (5) the first terms represent linear inertial, finite-Larmor-radius, and line-bending effects, while the last term on the left corresponds to the pseudogravitational term in some of the more phenomenological treatments. On the right-hand side, the first term represents the nonlinear interaction due to perturbations in the drift velocities, as discussed before for the case when $\delta V_{di} \ll \delta V_E$ (low ion temperature) in a slab model without shear and low β,[8] whereas the second term mainly represents the nonlinear effects of field curvature and shear (kink term). Equation (5) can be used as the starting point for nonlinear stability analysis of ballooning modes about any high-β tokamak equilibrium.

As a specific application we consider the case of axisymmetric concentric flux surfaces (first stability region and slightly beyond). We employ the quasimode representation[2]

$$\delta\Phi = \delta F(\psi,\chi)\exp\{in[\zeta - \int^\chi \nu(\chi',\psi)d\chi']\}, \qquad (6)$$

where ψ is the flux coordinate, χ and ζ are the poloidal and toroidal angles, respectively, and $\nu \equiv d\zeta/\delta\chi$ along a field line. In (6), the amplitude δF is slowly varying compared with the phase for large toroidal mode number. The linear part of (5) agrees with previously obtained results,[3,5] after substitution of (6):

$$\omega(\omega - n\omega_*)[1 + (s\chi - \alpha\sin\chi)^2]\delta F + k_c{}^2V_A{}^2(\partial/\partial\chi)\{[1 + (s\chi - \alpha\sin\chi)^2]\}\partial\delta F/\partial\chi$$
$$+ D[\cos\chi + (s\chi - \alpha\sin\chi)\sin\chi]\delta F = 0. \qquad (7)$$

Here, $\omega = \omega_r + i\gamma$ is the complex eigenfrequency, ω_* is the ion diamagnetic drift frequency, $s = d\ln q/d\ln r$ is the global shear, $k_c = 1/qR$ is the wave number of the background, α is the modulation of the shear ($\nu = \nu_0 - 2\alpha s\nu_0\cos\chi$), and $D \equiv -\kappa dP/\rho dr$. A more sophisticated representation, the ballooning formalism,[3] by which periodicity in toroidal angle is ensured, yields exactly the same equation (7), but with the domain of χ extended to the entire real axis.

An analytical dispersion equation can be obtained from the linear eigenvalue equation (7) in several ways, e.g., by expanding the eigenfunctions in a series of Hermite functions,[9] or, as carried out earlier[8] for drift-Alfvén waves in a slab, by Fourier transformation and subsequent Taylor expansion of the derivatives with respect to the poloidal mode number. That is, with $\hat{\varphi} \equiv \int_{-\infty}^{\infty}d\chi e^{im\chi}\delta F$, we have

$$\hat{\varphi}_m{}' \simeq \tfrac{1}{2}(\hat{\varphi}_{m+1} - \hat{\varphi}_{m-1}),\quad \hat{\varphi}_m{}'' \simeq \hat{\varphi}_{m+1} + \hat{\varphi}_{m-1} - 2\hat{\varphi}_m.$$

As a result of the poloidal variation of the driving force, all modes are linearly coupled to lower and upper sidebands,

$$a_m\hat{\varphi}_m + b_m{}^+\hat{\varphi}_{m+1} + b_m{}^-\hat{\varphi}_{m-1} = 0,$$

for all m, with, under neglect of terms $\propto\alpha^2$,

$$a_m \equiv (1+2s^2)[\omega(\omega - n\omega_*) - m^2k_c{}^2V_A{}^2] + 2Ds - \tfrac{1}{2}\alpha D,$$

$$b_m{}^\pm \equiv (\tfrac{1}{2} - s)D - \omega(\omega - n\omega_*)s^2 + (m^2 \pm m)k_c{}^2V_A{}^2s^2.$$

By use of the equation for $a_{m\pm1}\hat{\varphi}_{m\pm1}$, the sidebands can be expressed in terms of $\hat{\varphi}_m$ and $\hat{\varphi}_{m\pm2}$. By use of the equation for $a_{m\pm2}\hat{\varphi}_{m\pm2}$, $\hat{\varphi}_{m\pm2}$ can be expressed in terms of $\hat{\varphi}_{m\pm1}$ and $\hat{\varphi}_{m\pm3}$, after which the equation for $a_{m\pm1}\hat{\varphi}_{m\pm1}$ can be used to eliminate the newly appeared $\hat{\varphi}_{m\pm1}$, and so on. Thereby an expression for $\hat{\varphi}_m$ is obtained in terms of arbitrarily distant sidebands, which rapidly converges as the distance towards these sidebands is increased. Hence, the sidebands can be expressed in terms of the primary mode, and a dispersion equation is obtained:

$$a_m a_{m+1} a_{m-1} = \gamma_m^{+} b_m^{+} b_{m+1}^{-} a_{m-1} + \gamma_m^{-} b_m^{-} b_{m-1}^{+} a_{m+1},$$

where the coupling coefficients $\gamma_m^{\pm}$ have been defined as

$$\gamma_m^{\pm} = \left[1 - \frac{b_{m\pm1}^{\pm} b_{m\pm2}^{\mp}}{a_{m\pm1}a_{m\pm2}} - \frac{b_{m\pm1}^{\pm} b_{m\pm2}^{\mp} b_{m\pm2}^{\pm} b_{m\pm3}^{\mp}}{a_{m\pm1}a_{m\pm2}(a_{m\pm2}a_{m\pm3} - b_{m\pm2}^{\pm} b_{m\pm3}^{\mp})} + \dots\right]^{-1} \simeq \left(1 - \frac{b_{m\pm1}^{\pm} b_{m\pm2}^{\pm}}{a_{m\pm1}a_{m\pm2}}\right)^{-1}.$$

After an additional expansion in $D/(4k_c^2 V_A^2)$, the resulting dispersion relation reads

$$\omega = \tfrac{1}{2}n\omega_* \pm \left[\tfrac{1}{4}n^2\omega_*^2 + (m-1)^2 k_c^2 V_A^2 - \gamma_m^{-}\frac{D^2}{4k_c^2 V_A^2 m} + G_m(s)\right]^{1/2}, \qquad (8)$$

where $s \equiv d\ln q/d\ln r$ is the global shear and where, e.g., for $m=1$ (worst mode in view of minimal line bending) the function $G_m(s)$ is

$$G_m(s) = \frac{1}{k_c^2 V_A^2 (1+2s^2)^2}\left\{Ds(\gamma^- D - 2k_c^2 V_A^2) - \gamma^{-2}\frac{D^3}{4k_c^2 V_A^2}s^2(1-2s) + \gamma^- D^2 s^4 - R(s)\right\},$$

$$R(s) \equiv D\frac{2s^3}{1+2s^2}\left[2(1+2s^2)k_c^2 V_A^2 + \gamma^- D(1-2s)\right] - \gamma^+\left[2Ds - \gamma^-\frac{D^2}{4k_c^2 V_A^2}\left(1+2s^2 - \frac{4s^4}{1+2s^2}\right)\right]\left[D(\tfrac{1}{2}-s) + 2k_c^2 V_A^2 s^2\right]^2 \Big/ Q,$$

$$Q \equiv \left\{k_c^2 V_A^2 + \gamma^-\frac{D^2}{4k_c^2 V_A^2}\left[1 - \frac{4s^4}{(1+2s^2)^2}\right]\right\}(1+2s^2) - 2Ds.$$

Evidently, for small shear, destabilization prevails; for large shear, its overall effect is stabilizing. For example, the linear driving force is increased by $\frac{1}{6}$ of its value for $s=\frac{1}{2}$, reduced by $\frac{1}{2}$ for $s=1$ and exactly annihilated for $s=\infty$. This behavior is in agreement with previous results obtained differently.[2-5]

Returning to the nonlinear problem, we assume a harmonic time dependence in the nonlinear part of (5), which we may rewrite (using Maxwell's equations) as

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\vec{V}_{di}^{\,0}\cdot\nabla - V_A^2\nabla_\parallel^2\right)\Delta_\perp\hat{\Phi} - V_A^2\nabla_\parallel(\vec{e}_\parallel\cdot\Delta\nabla_\perp\hat{\Phi}) - 2\omega_{ci}(\vec{e}_\parallel\times\vec{\kappa})\cdot\nabla(\vec{V}_{de}^{\,0}\cdot\nabla\hat{\Phi})$$
$$= -\frac{\partial}{\partial t}\left\{\vec{e}_\parallel\times\nabla\left[\left(1+n_1\frac{\omega_*}{\omega_1}\right)\hat{\Phi}^{(1)}\right]\cdot\nabla\Delta_\perp\hat{\Phi}^{(2)} + \frac{V_A^2}{\omega_1\omega_2}\nabla(\vec{e}_\parallel\cdot\nabla\Delta_\perp\hat{\Phi}^{(1)})\cdot[\vec{e}_\parallel\times\nabla(\vec{e}_\parallel\cdot\nabla\hat{\Phi}^{(2)})]\right\} + (1\leftrightarrow2), \qquad (9)$$

where $(1\leftrightarrow2)$ denotes the previous terms with mode indices 1 and 2 interchanged. Here $\omega_* = -i\vec{V}_{di}\cdot\nabla^{(j)}$, where $\nabla^{(j)}$ operates on the phase, i.e., it gives a term $\propto n^{(1)}$. In order to study the nonlinear evolution in time, let us consider the Fourier components $\hat{\varphi}^m$ of $\hat{\varphi}^{(1)}$ with respect to the slow variation in poloidal angle.

First considering the case when $|\partial\ln\Phi_{nm}/\partial t| \ll \omega$ we may then drop the second-order time derivative in (9) if operating on the slowly varying amplitude. For the interaction of three modes satisfying the resonance conditions, i.e., $\boldsymbol{n} = \boldsymbol{n}_1 + n_2$, $m = m_1 + m_2$, $\Delta\omega = \omega - \omega_1 - \omega_2 \ll \omega_j$, $j = 1, 2$, we then obtain three coupled equations of the form

$$\partial\Phi/\partial t = c_{12}\Phi_1\Phi_2\exp(\pm i\Delta\omega t), \qquad (10)$$

where

$$c_{12} = i\frac{c}{B^0}\frac{\gamma(\psi,\chi)}{a}d_{12}(n_1\kappa_2 - n_2\kappa_1)(\omega_1+\omega_2)(2\omega - n\omega_*)^{-1},$$

with

$$d_{12} = (n_2^2 - \boldsymbol{n}_1^2)(1 - m_1 m_2 V_A^2/J^2\boldsymbol{B}^2\omega_1\omega_2) + n_1\boldsymbol{n}_2\omega_*(\boldsymbol{n}_2/\omega_1 - \boldsymbol{n}_1/\boldsymbol{\omega}_2).$$

Here, $-\gamma(\psi, \chi)$ is defined by the leading-order part of $\Delta_\perp \delta F \simeq -n^2\gamma\delta F$. In deriving (10) we have neglected the spatial variation of γ in the nonlinear terms. The factors κ_j are defined as the logarithmic derivative of Φ_{nm} with respect to radius. They originate from the operation ∇ on the slowly varying amplitude of $\Phi(j)$. Clearly this variation has to be taken into account, since otherwise the nonlinear terms vanish. Furthermore, in the case of marginal stability, the frequency is seen to satisfy $\partial\hat{D}/\partial\omega = 0$ (zero-energy wave), where $\hat{D}$ is the dispersion function [cf. (8)]. In this case the second-order time derivative of the amplitude cannot be neglected. We then arrive at a system of coupled mode equations where the second-order derivative enters in one equation, and only the first in the others. For a coupling of the zero-energy wave with two waves of the same energy sign we then have an explosively unstable system.[11] If the driving force is increased even further (linearly unstable case), the first time derivative will appear with an imaginary coefficient, together with the second time derivative, leading to a situation in which linear and explosively nonlinear growth enforce each other. We thus find several possibilities for nonlinear instability of ballooning modes.

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