Radiative-Recoil Corrections to Muonium Hyperfine Splitting

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This paper presents the results of an analytic calculation of radiative corrections to the leading recoil corrections in the muonium ground-state hyperfine splitting. The principal new results are of relative order $(\alpha/\pi)^2(m_e/m_\mu)\ln(m_\mu/m_e)$. Also presented is an estimate of the contribution of the hadronic term in the vacuum polarization. A contribution to the positronium ground-state hyperfine splitting of relative order $(\alpha/\pi)^2$, arising from vacuum polarization of exchanged photons, is also given.

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Higher-order contributions to the muonium ground-state hyperfine splitting (hfs) can be classified as recoil or nonrecoil, where the former are distinguished by the appearance of an additional power of the mass ratio m_e/m_μ. The present status of the recoil corrections is summarized in an accompanying paper[1] which also presents some new contributions. The leading ("one-loop") recoil corrections are of relative order $(\alpha/\pi)(m_e/m_\mu)\ln(m_\mu/m_e)$ with respect to the Fermi splitting E_F. The presence of the logarithm signals the fact that the dominant momentum region in the loop integration is $m_e \lesssim p \lesssim m_\mu$.

Although nonrecoil radiative corrections to the hfs have been known for a long time,[2] the importance of "radiative-recoil" corrections at the present level of accuracy was first pointed out by Caswell and Lepage.[3] They also calculated the leading contribution of relative order $(\alpha/\pi)^2 \times(m_e/m_\mu)\ln^2(m_\mu/m_e)$, which is due to vacuum polarization effects. In this note we report our calculation of the single logarithm terms of relative order $(\alpha/\pi)^2(m_e/m_\mu)\ln(m_\mu/m_e)$ arising from vacuum polarization of the exchanged photons and radiative corrections to the electron and muon lines.

Vacuum polarization is the most straightforward correction and we discuss it first. The leading contribution (relative order α^2) is associated with the one-loop exchange kernels in which the loop momentum is much greater than that in the wave function. The situation is illustrated in Fig. 1. For these contributions, the wave-function integrations may be decoupled and the momenta external to the kernels set equal to zero. Terms contributing to the hfs are easy to identify since they are associated with an even number of spatial γ matrices. If we represent the propagator of either of the exchanged photons by the spectral form

$$(\alpha/\pi)g_{\mu\nu}\int\rho(s)ds/(q^2-s) \tag{1}$$

and sum over ladder and cross-ladder exchanges, we find a splitting

$$\Delta E_{\rm VP}=\left(\frac{\alpha}{\pi}\right)^2 E_{\rm F}\,\frac{m_e m_\mu}{m_\mu^2-m_e^2}\int ds\,\rho(s)\int_0^1 d\lambda(2-\lambda)(8+\lambda)\left[\frac{m_\mu^2}{m_\mu^2\lambda^2+s(1-\lambda)}-\frac{m_e^2}{m_e^2\lambda^2+s(1-\lambda)}\right]. \tag{2}$$

This expression is exact to relative order α^2 and we note that it is symmetric under an interchange of the lepton masses. Nonrecoil contributions occur when loop momenta are of order of the electron mass. This can happen only for the electron loop contribution to vacuum polarization since the characteristic loop momentum is $\sqrt{s}$ in other cases. Allowing both electron and muon vacuum loop contributions in $\rho(s)$, expanding the resulting parameter integrals to $O(m_e/m_\mu)$, and discarding the nonrecoil term we find a splitting

$$\Delta E_{\rm VP(leptons)}=-\left(\frac{\alpha}{\pi}\right)^2\frac{m_e}{m_\mu}E_{\rm F}\left[2\ln^2\left(\frac{m_\mu}{m_e}\right)+\frac{8}{3}\ln\left(\frac{m_\mu}{m_e}\right)+\left(\frac{28}{9}+\frac{\pi^2}{3}\right)+O\left(\frac{m_e}{m_\mu}\right)\right] \tag{3}$$
$$=-6.61(1.0+0.250+0.113)\ \text{kHz}.$$

The double logarithm agrees with the result found previously by Caswell and Lepage[3] and our calculation obtains exact expressions for the singly logarithmic and constant terms. These coefficients agree with the results of a computation by Lepage.[4]

We have also estimated the hadronic contribution in (2) by assuming that the spectral function is dominated by the ρ resonance plus a 2π background. Using a Gounaris-Sakurai parametrization of the pion form factor[5] we find

$$E_{\mathrm{VP(pion)}}{}^{M} = 63.8(\alpha/\pi)^2(m_e m_\mu/m_\rho{}^2)E_{\mathrm{F}} = 0.14(2)\ \mathrm{kHz}. \qquad (4a)$$

Contributions due to the ω and φ resonances ($\sim$0.03 kHz) plus higher mass contributions (which we estimate very roughly as 0.05 kHz), should be added to this. Then the total hadronic contribution is

$$\Delta E_{\mathrm{VP(hadronic)}}{}^{M} = 0.22(4)\ \mathrm{kHz}. \qquad (4b)$$

The uncertainty reflects the sensitivity of this result to the values of the fitting parameters. In any case the hadronic correction is small.

In (2) we may take the limit $m_\mu \to m_e$ to calculate the shift in the positronium hfs due to vacuum polarization. For this case we need only retain the term with an electron in the vacuum loop since the contribution from more massive loop particles is suppressed by a factor $(m_e/m_{\mathrm{loop}})^2$. Then we find

$$\Delta E_{\mathrm{VP}}{}^{\mathrm{Ps}} = \tfrac{5}{3}(\alpha/\pi)^2 E_{\mathrm{F}}{}^{\mathrm{Ps}} = 1.05\ \mathrm{MHz}. \qquad (5)$$

Radiative corrections to the electron line are shown in Fig. 2. The justification of these as the correct set is complicated by the possibility that diagrams in which the radiative photon spans an arbitrary number of Coulomb exchanges between the external lines might also be important. This would occur for a given diagram in the Feynman gauge because the associated loop integrations peak at momenta of order $\alpha^2 m_e$ and so generate inverse powers of α. These can compensate the additional powers of the coupling constant. By adopting a special gauge, the Fried-Yennie gauge,[6] it can be shown diagram by diagram that kernels involving additional Coulomb rungs contribute only in higher order. This gauge, however, must be used with some care near the mass shell since both the electron propagator and vertex have rapid but finite variations in that region.[7] To regulate this behavior we keep the external electron leg slightly off the mass shell during the calculation and put the external three-momenta equal to zero only at the end. In that case we find that the infrared pieces of the spanning photon and external electron self-energy cancel exactly. It is also possible in this way to justify the "scattering approximation" in which the external lines are initially put on the mass shell, and the infrared divergence is instead regulated with a photon mass. Then since the set of diagrams in Fig. 2 is gauge invariant, we may calculate in any gauge.

In addition to the infrared region the ultraviolet region requires a few remarks. Because of the Ward identity the divergences in the vertices cancel against those in the electron self-energies. As usual, proper counting is provided by including only one external self-energy. The leading double logarithms here and in the vacuum polarization are associated with these divergent renormalizations. It is not surprising that the electron line contributions cancel as found previously by Caswell and Lepage.[3] This is in contrast to the situation with vacuum polarization where the divergence is removed by renormalization rather than by cancellation. But the presence of double

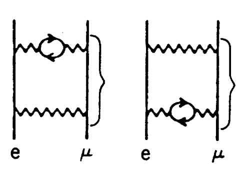

FIG. 1. Vacuum polarization corrections to the one-loop ladder kernel. Note that the vacuum loop can circulate leptons or hadrons. The brace on the muon lines means that crossed diagrams should be added.

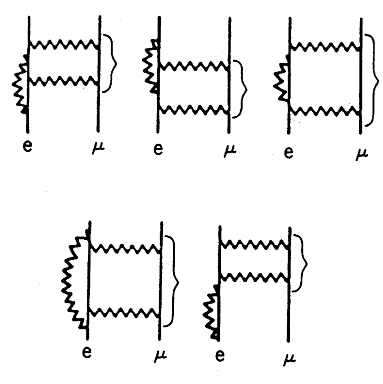

FIG. 2. Radiative corrections to the electron line in the one-loop kernels. For proper renormalization, only one external electron self-energy is needed.

logarithms in the various diagrams complicates the calculation of the nonleading pieces. We have found that by rearranging and combining terms from the momentum-space expression for each diagram the double logarithms can be explicitly cancelled before integration. In addition, while different diagrams give different denominator structures, further cancellations take place and the logarithmic terms can be evaluated easily as multiples of the lowest-order kernel. The result is

$$E_{e\text{-}line}{}^{M} = \frac{15}{4}\left(\frac{\alpha}{\pi}\right)^2 \frac{m_e}{m_\mu} E_F \ln\left(\frac{m_\mu}{m_e}\right) = 2.32 \text{ kHz}. \quad (6)$$

The muon radiative corrections require different approximations from those used for the electron. This is because we are concerned with exchanged momenta that are large compared with m_e, but small compared with m_μ. Thus for electron radiative corrections we neglect m_e wherever possible, while for muon radiative corrections we neglect the exchanged momentum wherever possible. Thus suggests the use of the famous low-energy theorem for Compton scattering.[8] Unfortunately this theorem applies to the scattering of real photons on a nonelectromagnetic target, so that possible infrared divergences are not considered. Nevertheless, we have been able to show, using formal operator techniques,[9] that an analogous result holds for our case. Consequently the leading corrections to order q/m_μ can be incorporated in an $O(\alpha)$ anomalous-moment term in the vertex. Replacing each muon vertex by

$$\gamma_\mu \to \gamma_\mu + (\alpha/4\pi m_\mu)\not{q}\gamma_\mu,$$

we find that the muon factor to relative order α becomes

$$\left\{-\left(1+\frac{\alpha}{2\pi}\right)\gamma_\beta\not{q}\gamma_\alpha + \frac{\alpha}{2\pi} q_0\gamma_\beta\gamma_\alpha\right\}\left\{\frac{1}{q^2-2m_\mu q_0+i\epsilon} + \frac{1}{q^2+2m_\mu q_0+i\epsilon}\right\}. \quad (7)$$

Taken together, the logarithms arising from the two $\alpha/2\pi$ terms cancel. This is an old result.[10] What is new in our work is an argument that all other radiative corrections associated with the muon fail to contribute to the logarithm.

Even though the muon-line radiative corrections do not yield logarithms directly, it is conventional to rearrange the expression so that a logarithmic radiative-recoil correction does appear. This is because the dominant nonrecoil contributions to the hfs contain the total muon magnetic moment, an experimentally measured parameter, as an overall factor. This appears in Eq. (3) of Ref. 1 where the total muon magnetic moment $[\mu_\mu = (e\hbar/2m_\mu c)(1+a_\mu)]$ occurs in E_F. Since the experimentally determined parameter is μ_μ rather than m_μ, comparisons between theory and experiment are conventionally made in this way. The term $1+\alpha/2\pi$ in (7) provides the justification of this factorization to order α since it multiplies the same expression that would have occurred without muon radiative corrections. (Justification of the higher orders in α is fairly obvious, because recoil may be neglected to present orders of interest.) With this convention, the second $\alpha/2\pi$ term in (7) is viewed as an additional radiative-recoil correction:

$$E_{\mu\text{-}line}{}^{M} = \tfrac{3}{2}(\alpha/\pi)^2(m_e/m_\mu)E_F \ln(m_\mu/m_e) = 0.93 \text{ kHz}. \quad (8)$$

We summarize the results by giving the expression for $Q(m_\mu/m_e)$ in Eq. (3) of Ref. 1:

$$Q = -2\ln^2\left(\frac{m_\mu}{m_e}\right) + \frac{31}{12}\ln\left(\frac{m_\mu}{m_e}\right) - \frac{28}{9} - \frac{\pi^2}{3} + 1.9 \quad (9)$$

(the 1.9 comes from our estimate of the hadronic vacuum polarization term). The new contribution from (9) (i.e., excluding the first term) yields a net value of 1.1 kHz; the total contribution is −5.5 kHz. It is hard to estimate the size of the uncalculated nonlogarithmic contributions which arise from radiative corrections in the electron and muon lines. In obtaining (6) and (8), we have set aside many such contributions. In principle, they all could be calculated analytically, but it will probably be better to evaluate them numerically when the need arises. Their contribution will be $\lambda(\alpha/\pi)^2(m_e/m_\mu)E_F = (0.116\lambda)$ kHz where we expect λ to be a number of order unity, but easily as large as 3 or 4. We take the uncertainty due to these missing terms to be 0.5 kHz. The hadronic contribution (4) is of the order of the experimental uncertainty in the measurement of the muonium hfs, but is still small compared to the uncertainty due to the measurement of the muon magnetic moment and to the size of various uncalculated terms. These results are incorporated in the comparison with experiment in the accompanying paper.[1]

During the course of this work we have benefitted greatly from constant interactions with G. Peter Lepage. Our confidence in the vacuum polarization results was greatly enhanced by his confirmatory numerical work. He expects to apply the same techniques to the electron- and muon-line contributions and perhaps eliminate the uncertainties there. We also acknowledge useful discussions with G. T. Bodwin, T. Kinoshita, and G. W. Erickson. This work was supported by the National Science Foundation.

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