Corrections to the Muonium Hyperfine Splitting of Relative Order $\alpha^2(m_e/m_\mu)$

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This paper presents the results of an analytic calculation of the corrections of relative order $\alpha^2(m_e/m_\mu)$ to the muonium ground-state hyperfine splitting due to exchanged photons. Theory and experiment are compared, with these corrections and some radiative-recoil contributions described in an accompanying paper taken into account.

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The comparison of the measured hyperfine splitting (hfs)[1] in the muonium ground state with the quantum electrodynamics (QED) prediction[2] currently provides the most stringent test of relativistic two-body bound-state theory. This paper describes the calculation of the recoil corrections (excluding radiative corrections) of relative order $\alpha^2(m_e/m_\mu)$. (Here α is the fine-structure constant and m_e and m_μ are the electron and muon masses.) The $\ln\alpha^{-1}$ terms in order $\alpha^2(m_e/m_\mu)$ have been calculated previously[3,4] and $\ln(m_\mu/m_e)$ terms have been shown to vanish.[5] We present here the results of an analytic calculation of the remaining nonlogarithmic contributions. The so-called radiative-recoil corrections are described in an accompanying paper.[6] Complete details of both will be given in papers in preparation.

In the Bethe-Salpeter approach, the energy levels of a two-particle bound state are given by the positions of the poles in the four-point function. Since the Coulomb potential dominates the QED bound-state problem, we start with an analysis of the Coulomb ladder. Each loop of the ladder is separated into two pieces. One piece, which we call the "nonrelativistic loop," is treated exactly and leads to a three-dimensional wave equation whose solution yields the lowest-order wave functions and energy levels. The other piece, which we call the "remainder loop," is treated perturbatively along with the various two-particle irreducible parts of the four-point graphs. There are infinitely many ways to partition the Coulomb ladder loops into a lowest-order part and perturbative part and, indeed, a variety of approaches have been presented in the literature.[7] In general, one must include at least the Coulomb-Schrödinger part of each loop in lowest order, since this part is essentially nonperturbative.

For muonium, it is convenient to take advantage of the small mass ratio ($m_e/m_\mu \ll 1$) to incorporate more physics into the lowest-order problem. By requiring the muon to be on its positive-energy mass shell, and setting aside some terms involving m_μ^{-1}, one arrives at the Gross equation[8] for the bound-state wave functions,

$$(H_e+\tilde{V})\psi_n = E_n'\psi_n, \tag{1}$$

where $H_e=\vec{\alpha}_e\cdot\vec{p}+\beta_e m_e$ is the Dirac Hamiltonian, $\tilde{V}=m_2V_c/E$ where V_c is the Coulomb potential (other possibilities exist), ψ_n includes a factor of $\frac{1}{2}(1+\beta_\mu)$, and E' is related to the total energy E through $E'=(E^2-m_\mu^2+m_e^2)/2E$. Obviously, Eq. (1) incorporates the relativistic properties of the electron, while most of the dynamics of the muon have been suppressed except for effects associated with the reduced mass.

The energy shifts through second order in the perturbation are[4]

$$\Delta E' = \langle 0|\tilde{K}|0\rangle\left\{1+\left\langle 0\left|\left[\frac{\partial\tilde{V}}{\partial E'}+\frac{\partial\tilde{K}}{\partial E'}\right]\right|0\right\rangle\right\}+\sum_{n\neq 0}\frac{\langle 0|\tilde{K}|n\rangle\langle n|\tilde{K}|0\rangle}{E_0'-E_n'}, \tag{2}$$

where $\tilde{K}$ is the sum of perturbation kernels, examples of which are shown in Fig. 1. In carrying out the calculation, it is sometimes convenient to remove certain pieces of the first-order terms and combine them with second-order terms. Here we shall describe the main lines of the calculation without attempting to detail these refinements.

First we discuss the first order in $\tilde{K}$. Any given kernel contains many orders in α and m_e/m_μ; but in general, as the number of loops excluding "nonrelativistic loops" increases, the least order in α increases. For the present work, it turns out that we need up to two such loops. We can use the propagator decomposition described earlier and the Gross equation to rearrange the kernels. In this way we organize the kernels so that there are no more than two loops and no "remainder loops." The final set of kernels is shown in Fig. 1. Note that the subtracted one-loop kernels have the effect of removing lower-order contributions that are over counted in the two-loop kernels. For example, each of the kernels in the first line of Fig. 1 contains leading-order (Fermi splitting) contributions; altogether, the leading order appears exactly once.

It is interesting that one arrives at the set of kernels shown in Fig. 1 regardless of the choice of wave equation.[9] Different choices of wave position manifest themselves as different decompositions into lowest- and second-order contributions in Eq. (2). That is, any feature not incorporated in the wave function is restored in the

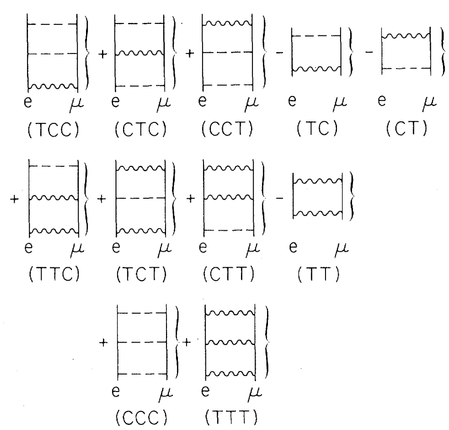

FIG. 1. Kernels contributing to the muonium hfs to the order of interest. Dashed lines represent Coulomb interactions (C) and wavy lines represent transverse photons (T). The brace indicates that photon lines are to be inserted in the muon leg in all possible ways. The labeling in parentheses indicates the order of attachment of the photon lines to the electron leg.

second-order terms. The use of the Gross equation means that the relativistic properties of the electron and certain recoil corrections are in the wave function rather than in the sum over states.

In carrying out the evaluation of the first-order perturbation theory matrix elements, it is useful to have covariant denominators for all of the photon propagators, rather than the awkward noncovariant denominators contained in the Coulomb photon exchanges C and the transverse photon exchanges T. In order to accomplish this, we carry out a transformation *within* the kernels to the covariant Feynman gauge. It is still convenient to distinguish photons with spatial indices (denoted by V) from those with temporal indices (denoted by O). The net effect of the gauge transformation on the kernels is that $T \rightarrow V$ and $C \rightarrow O$ for the set shown in Fig. 1. There are, in addition, residual gauge terms associated with the external fermion lines. Those associated with the muon line vanish because the muon is on the mass shell in the Gross equation. Those associated with the electron line give contributions that tend to be smaller than the order of interest, partly because of cancellations between the two-loop and one-loop kernels. Gauge terms arising from the graphs labeled (TCT) do contribute in relative order $\alpha^2(m_e/m_\mu)$. However, these contributions are precisely canceled by some terms of second order in $\tilde{K}$.

At this stage we wish to expand the first-order energy shifts in powers of m_e/m_μ. Thus, it is tempting simply to expand the muon factor of each graph in powers of $1/m_\mu$. However, the presence of previously calculated terms of relative order $\alpha(m_e/m_\mu)\ln(m_\mu/m_e)$ shows that this is not possible.[10] We organize the calculation so that these terms, as well as the leading-order contributions, can be identified and extracted at the start. We also note that individual graphs contain spurious nonrecoil contributions. These, as well as spurious $\alpha^2(m_e/m_\mu)\ln(m_\mu/m_e)$ terms,[11] are eliminated before integration by combining kernels generated by permuting photon connections on the muon leg (see Fig. 1). Thus, we are finally able to make a direct m_μ^{-1} expansion before integration.

Having carried out these procedures, we find that in the two-loop contributions the muon factor always contains at least one δ function of the time component of momentum. For most of the contributions we can, in the order of interest, neglect the dependence of the kernels on the momentum variables of the wave function. Then we are left with certain seven-dimensional integrals to evaluate. We treat these by combining denominators using Feynman parameters and carrying out the momentum integrations. We are then able to construct six-dimensional integrals involving only nonrelativistic propagators that lead to identical parameter integrals. These six-dimensional integrals can be evaluated easily by Fourier transformation to coordinate space. There are some cases in which we must retain the wave-function momentum dependence in the kernels. However, it is then legitimate to make nonrelativistic approximations, which eliminate the time components of the momentum integrations. The nine-dimensional integrals that arise in this way are relatively straightforward to evaluate by complex integration. The twelve-dimensional integral associated with the kernel VOV can be computed by use of a variant of the method of Dalgarno and Lewis.[12]

Finally, let us discuss the calculation of the second-order energy shifts. In second order in $\tilde{K}$ at least one of the kernels must involve the hyperfine interaction, but one may be spin independent. A spin-independent contribution does arise from Coulomb-potential remainder loops and from the convection part of a transverse-photon interaction. Because of the structure of $\tilde{K}$, the contribution where the hyperfine interaction is taken twice in the sum over states has two or more Coulomb interactions between hyperfine interactions. It can be worked out easily by using the Dalgarno-Lewis method. The result, including contributions from the $\partial\tilde{K}/\partial E'$ term in Eq. (2), agrees with a calculation of Caswell and Lepage.[13] The contributions involving a spin-independent factor combine naturally with certain terms from the first-order contribution. The resulting sum over states is also evaluated by using the Dalgarno-Lewis method.

The theoretical expression for the muonium hfs is[2]

$$\Delta\nu = E_F\left\{1 + a_e + \alpha^2 R(\alpha) + \tfrac{3}{2}\alpha^2 - 3\frac{\alpha}{\pi}\frac{m_R}{m_\mu - m_e}\ln\frac{m_\mu}{m_e} + \alpha^2\frac{m_R}{m_e + m_\mu}\left[2\ln\alpha^{-1} \underline{- 8\ln 2 + 3\tfrac{11}{18}}\right] + \left(\frac{\alpha}{\pi}\right)^2\frac{m_e}{m_\mu}Q\left(\frac{m_\mu}{m_e}\right)\right\}, \quad (3)$$

where $m_R = m_e m_\mu/(m_e + m_\mu)$ and

$$E_F = \frac{\frac{8}{3}\alpha^2 m_R^3(1 + a_\mu)}{m_e m_\mu} = \tfrac{16}{3}\alpha^2 c R_\infty\left(\frac{m_R^3}{m_e m_\mu}\right)(1 + a_\mu);$$

a_e and a_μ are the electron and muon magnetic moment anomalies. R incorporates additional radiative corrections[14] and Q contains the radiative-recoil corrections given in an accompanying paper.[6] The other terms were derived by the unified treatment described here, the new contributions being underlined. Because we have expanded in powers of m_μ^{-1}, the new term is not applicable to positronium. Its contribution to muonium is -2.2 kHz. Those arising from Q have a net value of $-5.5(0.5)$ kHz of which 1.1 kHz is a new contribution. Using the ac Josephson value[15] for α, we find

$$\Delta\nu(\text{theory}) = 4\,463\,303.3(1.7)(3.0).$$

The experimental uncertainty of 1.7 results from the uncertainties due to m_μ (1.4 kHz) and α (1.0 kHz). The present work has reduced the theoretical uncertainty from 5.0 to 3.0 kHz by improving the recoil corrections. The remaining 3.0 kHz arises mainly from $R(\alpha)$.[14] This value is in good agreement with the experimental result[1]

$$\Delta\nu(\text{expt}) = 4\,463\,302.88(0.16)\text{ kHz}.$$

Since the theory prediction uses as input the value of the fine-structure constant α, one can regard the measurement of the muonium hfs as a means for determining α. Recent refinements in the theory of the electron anomalous magnetic moment have resulted in greater precision in the determination of α from pure elementary-particle physics.[16] Comparison of these results could, when the theoretical uncertainties in the muonium hfs are reduced further, give bounds on the scales of internal electron structure purely from particle physics measurements.[16] Comparison of α from the muonium hfs with condensed-matter determinations provides an important test of the internal consistency of QED, as well as of our understanding of the theory of the condensed matter measurements. We list below the values of α determined from the muonium hfs,[1] the electron anomalous moment,[16] the ac Josephson effect,[15] the quantum Hall effect,[17] and a combination of the quantum Hall and ac Josephson measurements[17]:

$$\alpha^{-1}(\text{muonium hfs}) = 137.035\,969(21)(46)$$

$$\alpha^{-1}(\text{anomalous moment}) = 137.035\,993(5)(9)$$

$$\alpha^{-1}(\text{ac Josephson}) = 137.035\,963(15)(?)$$

$$\alpha^{-1}(\text{quantum Hall}) = 137.035\,968(23)(?)$$

$$\alpha^{-1}(\text{ac-Jos. and qu. Hall}) = 137.035\,965(12)(?).$$

The first errors listed are experimental and the second theoretical. The question mark in the condensed-matter determinations indicates that the theoretical uncertainties are unknown; they are possibly very small in comparison to the experimental ones. There is no clear discrepancy between these results, but further reductions in the errors and a better theoretical understanding of the condensed-matter measurements are clearly desirable.

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$$[1+(\vec{\alpha}_e\cdot\vec{p}/2m_e)][1-(\vec{\alpha}_\mu\cdot\vec{p}/2m_\mu)]\psi_S,$$

where ψ_S is the Schrödinger wave function.

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