serve that the Totsuji-Ichimura convolution approximation[7] usually considered for the homogeneous OCP but readily extended to the general case,

$$\rho^T(q_1q_2q_3) = \rho^T(q_1q_2)\rho^T(q_2q_3)/\rho(q_2) + \rho^T(q_1q_3)\rho^T(q_3q_2)/\rho(q_3) + \rho^T(q_3q_1)\rho^T(q_1q_2)/\rho(q_1) + \int dq_4\,\rho^T(q_1q_4)\rho^T(q_2q_4)\rho^T(q_3q_4)/\rho^2(q_4), \tag{13}$$

does indeed satisfy $M(l;2)=0$ whenever $M(l;1)=0$. This is perhaps not surprising since (13) is correct to first order in the plasma coupling parameter but may be responsible for the good results one obtains with this approximation[7] and should be preserved in modifications designed to improve its short-distance behavior.

It appears, rather surprisingly, that when $\mathfrak{D}$ is equal to the half-space, i.e., $r^1>0$, then correlations "parallel to the wall" decay like $r_\parallel^{-\nu}$. This can be verified explicitly for the OCP in $\nu=2$ at $\beta e^2=2$ and perturbationally in the general case.[4] An extension of our theorem shows that this is sufficient for the $l=0$ sum rule but not for $l>0$.[8] Indeed we argue[3,8] that stronger decay which would imply the $l=1$ sum rule would have some very unphysical consequences.

This work was supported in part by the National Science Foundation under Grant No. CHE 80-01969, the U. S. Office of Naval Research under Grant No. N-00014-81-C-0776, and the U. S. Air Force Office of Scientific Research under Grant No. 82-0016.

[1]Ch. Gruber, Ch. Lugrin, and Ph. A. Martin, J. Stat. Phys. 22, 193 (1980).

[2]Ch. Gruber, J. L. Lebowitz, and Ph. A. Martin, J. Chem. Phys. 75, 994 (1981).

[3]L. Blum, D. Henderson, J. L. Lebowitz, Ch. Gruber, and Ph. A. Martin, J. Chem. Phys. 75, 5974 (1981).

[4]B. Jancovici, Phys. Rev. Lett. 46, 386 (1981), and J. Stat. Phys. 28, 43 (1982), and to be published.

[5]D. Brydges and P. Federbush, Commun. Math. Phys. 73, 197 (1980).

[6]M. Baus and J. P. Hansen, Phys. Rev. 59, 1 (1980).

[7]H. Totsuji and S. Ichimaru, Prog. Theor. Phys. 50, 753 (1973); S. Ichimaru, to be published.

[8]L. Blum, Ch. Gruber, D. Henderson, J. L. Lebowitz, and Ph. Martin, to be published.

Fine Structure of Phase Locking

Leon Glass[(a)]
Department of Physiology, McGill University, Montreal, Quebec H3G 1Y6, Canada

and

Rafael Perez
Instituto de Física, Universidad Nacional de México, Mexico City, Mexico

(Received 9 April 1982)

A simple mathematical model is given which shows how phase locking, bistability, period-doubling bifurcations, and chaos may result from periodic stimulation of nonlinear oscillators. A new fixed-point theorem, which extends the classic results of Arnold, is used in the analysis.

PACS numbers: 05.40.+j, 03.40.-t, 87.10.+e

More than fifty years ago, in a study of electric circuits representing coupled pacemaker sites of the heart, it was demonstrated that as the frequency of periodic input to a nonlinear oscillator was changed, many types of phase-locked rhythms mimicking normal and pathologic cardiac rhythms could be observed.[1] Subsequent studies showed that periodic inputs to nonlinear oscillators could also lead to bistability (in which one of two different phase-locked patterns was observed, depending on the initial condition) and aperiodic dynamics.[2,3] Recently, period-doubling bifurcations and aperiodic "chaotic" dynamics were observed from periodically driven nonlinear oscillators.[4-7] The transition from periodic to aperiodic dynamics displays universal properties predicted theoretically.[8,9]

Our interest in phase locking stems from studies of the effects of periodic stimulation of spontaneously beating cardiac cells by brief electrical current pulses. In response to periodic input, phase locking, period-doubling bifurcations, and aperiodic dynamics were experimentally observed as the frequency and amplitude of the stimulus were varied.[5] For situations such as the cardiac oscillation, in which the oscillator relaxes rapidly back to the limit cycle following a perturbation, the dynamics can be approximately described by the finite difference equation

$$x_{i+1} = f(x_i, \tau) = g(x_i) + \tau, \tag{1}$$

where $x_i \pmod 1$ is the phase of the oscillation immediately before the ith stimulus, τ is the period of the stimulation, and g is an experimentally measured function which describes the effect of a single pulse on the rhythm. The phase $x_i \pmod 1$ is defined on the unit circle, and τ is measured in units of the intrinsic period of the oscillator without perturbation. The function f is called the Poincaré map. As an example, we analyze

$$x_{i+1} = x_i + \tau + b \sin 2\pi x_i, \tag{2}$$

which has been proposed as a mathematical model for periodically forced nonlinear oscillators.[7] If the Poincaré map is a monotonic function of x [e.g., for $0 < b \leq 1/2\pi$ in Eq. (2)], the dynamics are well understood.[10] For $b > 1/2\pi$ in Eq. (2), the Poincaré map is not monotonic and bistability, cascading period-doubling bifurcations, and chaos have been observed.[11,12] Experimental studies of the cardiac oscillator show that for some stimulus strengths, the Poincaré map is not monotonic [Ref. 5, Fig. 2(d)]. In the following, a new topological result and numerical computations are used to analyze Eqs. (1) and (2) as the Poincaré map changes from a monotonic to a nonmonotonic function. The analysis shows that if the Poincaré map is not monotonic, both bistabiltiy and period-doubling bifurcations are observed over a large range of values in parameter space.

Iteration of Eq. (1) generates a sequence of points x_0, $x_1 = f(x)$, $x_2 = f(x_1) = f^2(x_0), \ldots$. There is a fixed point of period N if

$$x_N = x_0; \quad x_i \neq x_0, \quad \text{for } i = 1, 2, \ldots, N-1. \tag{3}$$

If there is a fixed point of period N, then there will be a cycle of period N, x_0^*, $x_1^*, \ldots, x_N^*$, where $x_N^* \pmod 1 = x_0^* \pmod 1$. A cycle is (locally) stable if

$$\prod_{i=0}^{N-1} \left| \left(\frac{\partial f}{\partial x_i} \right)_{x_i = x_i^*} \right| < 1. \tag{4}$$

Stable cycles of period N are associated with stable phase-locked dynamics.[3,5-7] If there is a stable cycle of period N, there is stable $N{:}M$ phase locking where $M = x_N^* - x_0^*$. The ratio, $\rho = M/N$, is called the rotation number.[10] If an extremum of f is on a cycle, the cycle is stable and is called a superstable cycle.[9] We call the locus of superstable cycles as a function of b and τ the skeleton of the phase-locking zones. The skeletons of two-parameter quartic and cubic maps have been described.[13]

The degree (or winding number) of a function, $f(x)$, defined on the unit circle measures the number of times the function winds around the unit circle as x traverses the unit circle once. In the following, we assume that the degree of $g(x)$ in Eq. (1) is 1 so that

$$g(x+j) = g(x) + j, \tag{5}$$

where j is an integer. Equations (3) and (5) lead to a translational symmetry in the phase-locking zones. If there is $N{:}M$ phase locking for a given value of τ, then for $\tau' = \tau + p$, where p is an integer, there will be $N{:}(M + pN)$ phase locking.[6] If in addition $g(x)$ is odd $[g(x) = -g(-x)]$ and if there is stable $N{:}M$ phase locking for an initial condition x_0^* with $\tau = p + \epsilon$ in Eq. (1), then there will also be stable $N{:}[(2p+1)N - M]$ phase locking for an initial condition of $-x_0^*$ with $\tau' = p + 1 - \epsilon$.[6]

If the Poincaré map, Eq. (1), is monotonic, ρ is a monotonic function of τ which is independent of the initial condition and is piecewise constant on the rationals.[10] Consequently phase-locking zones corresponding to all rational ratios are present.[10] What happens to these zones when the Poincaré map is not monotonic?

Assume that $g(x)$ is of degree 1 and has a single maximum, x_{max}, and minimum, x_{min}, on the interval $[0,1]$. Let $x_i = H_i(\tau)$ where the $H_i(\tau)$ are functions found by iterating Eq. (1) from $x_0 = x_{max}$. From Eqs. (1) and (5)

$$H_N(j) - H_N(j-1) = N. \tag{6}$$

There will be a superstable cycle for each value of τ for which $x_0 \pmod 1 = H_N(\tau) \pmod 1$. Since $H_N(\tau) \pmod 1$ must equal any fixed value between 0 and 1 at least N times as τ varies from $j-1$ to j, there will be a minimum of N superstable cycles associated with N different rotation numbers occurring at N distinct values of τ. Figure 1 shows $H_i(\tau)$, $i = 1, 2, 3$ for three values of b as a function of τ for Eq. (2). The iterates of the minimum also give rise to stable cycles. Therefore, from the translational symmetry of phase-locking

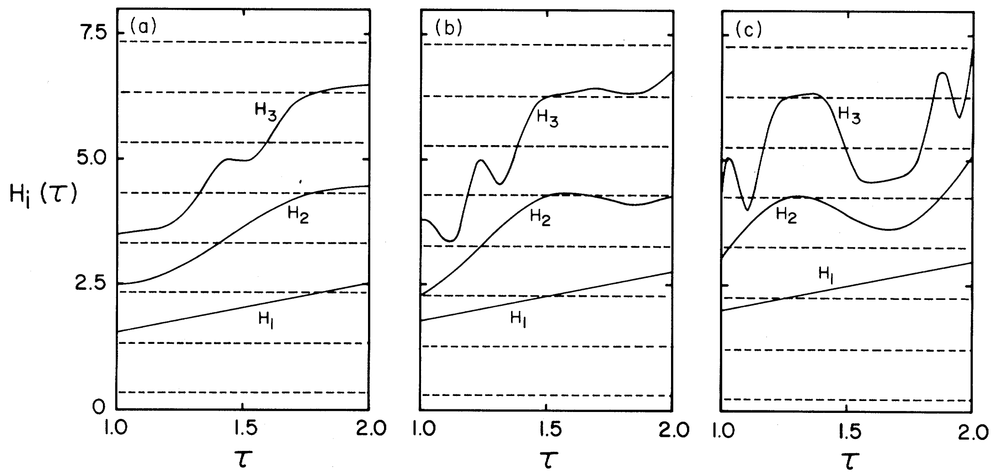

FIG. 1. The functions $H_i(\tau) = f^i(x_{max})$ for Eq. (2), in which $x_{max} = (1/2\pi)\cos^{-1}(-1/2\pi b)$. $H_i(\tau)(\mathrm{mod}\ 1) = x_{max}$ at the intersections with the dashed horizontal lines. These intersections give the values of b and τ at the superstable cycles. (a) $b = 0.25$; (b) $b = 0.50$; (c) $b = 0.75$.

zones, for a Poincaré map, Eq. (1), of degree 1 with a single maximum and minimum on the interval *(0, 1), there will be at least two values of τ at which there exist superstable cycles for each rational rotation number.*
theorem is the main analytic result of this Letter.

To illustrate this result, consider Eq. (2). The boundaries of the stable N:M phase-locking zones ($1 \leq N \leq 5$) for $0 \leq b \leq 1/2\pi$ in Eq. (2) shown in Fig. 2 agree with Arnold's classic results.[10] For $b > 1/2\pi$, the right-hand side of Eq. (2) has a maximum and a minimum, and therefore there must be at least two values of τ that give superstable cycles for each rational rotation number. The skeleton for $b > 1/2\pi$ corresponding to the stable phase-locking zones for $0 < b < 1/2\pi$ is shown in Fig. 2.

Next consider the period-doubling bifurcations for Eq. (2). For $\tau = 1$ as b increases there is a period-doubling bifurcation to a single orbit of period 2 at $b = \pi^{-1}$, a bifurcation to two stable period-2 orbits at $b = 0.5$, a bifurcation to two stable period-4 orbits at $b = (0.25 + 1/2\pi^2)^{1/2}$, and additional bifurcations leading to chaos as b continues to increase.[11,12] Sequences of period-doubling bifurcations also originate from other phase-locking zones.[12] The geometry of the period-doubling zones has been investigated by numerically computing the skeleton. In Fig. 3 all the superstable cycles associated with cycles for which $\rho = 1$ are shown for $1 \leq N \leq 4$, $b < 0.65$, and $0.7 < \tau < 1.3$. On the lines $b = \tau - 0.4$, $b = 1.4 - \tau$ in Fig. 3 there is a sequence of bifurcations which follows the Sarkovski sequence.[9] Further, a topologically equivalent skeleton appears to arise in other V-shaped regions tangent to $b = 1/2\pi$ in

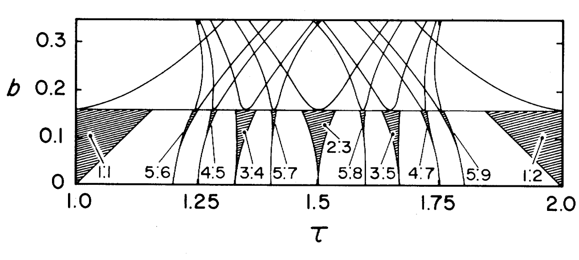

FIG. 2. The $N:M$ phase-locking regions for $1 \leq N \leq 5$ for $b < 1/2\pi$ and the associated superstable cycles for $b > 1/2\pi$. There is bistability at the intersections of the loci of the superstable cycles.

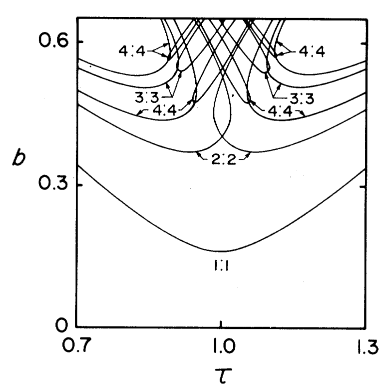

FIG. 3. $N:M$ superstable cycles for ($1 \leq N = M \leq 4$) associated with phase-locking patterns with rotation number, $\rho = 1$.

Fig. 2. A topologically equivalent skeleton for period-1, -2, and -3 orbits of the asymmetric cubic map has been found.[13]

On the basis of the preceding results we propose the following structure for the skeleton and phase-locking zones of Eq. (2). Each zone of stable phase locking for $b < 1/2\pi$ extends through $b = 1/2\pi$ and then splits into two branches (Fig. 2). In the V-shaped region of the extensions of each "Arnold tongue" are period-doubling bifurcations. The skeleton of the phase-locking zones in each one of these V-shaped regions is topologically equivalent to the skeleton for $\rho = 1$, Fig. 3, but with a different rotation number. Related conjectures for a two-parameter family of quadratic maps of the plane have recently appeared.[14]

If, as many have proposed,[3,5-7,10-12] periodically forced nonlinear oscillators can be modeled by one-dimensional finite difference equations having the topological and symmetry characteristics of Eqs. (1) and (2), then the topological structure of the phase-locking zones shown in Figs. 2 and 3 may be widely observed in the biological and physical sciences. Experimental observations will be difficult since many of the phase-locking zones only occupy small areas of the parameter space, and not all initial conditions necessarily attract to the stable phase locking, even when it is present.

This research has been supported by grants from the Natural Sciences and Engineering Research Council of Canada and the Canadian Heart Association. We thank M. Guevara, M. C. Mackey, and J. Guckenheimer for helpful conversations.

[(a)]Corresponding author.

[1]B. Van der Pol and J. Van der Mark, Philos. Mag. 6, 763 (1928).

[2]M. L. Cartwright and J. E. Littlewood, J. London Math. Soc. 20, 180 (1945); N. Levinson, Ann. Math. 50, 127 (1949); C. Hayashi, *Nonlinear Oscillations in Physical Systems* (McGraw-Hill, New York, 1964).

[3]M. Levi, Mem. Am. Math. Soc. 32, No. 244 (1981).

[4]K. Tomita and T. Kai, J. Stat. Phys. 21, 65 (1979); P. S. Linsay, Phys. Rev. Lett. 47, 1349 (1981); J. Testa, J. Pérez, and C. Jeffries, Phys. Rev. Lett. 48, 714 (1982).

[5]M. R. Guevara, L. Glass, and A. Shrier, Science 214, 1350 (1981).

[6]M. R. Guevara and L. Glass, J. Math. Biol. 14, 1 (1982).

[7]G. M. Zaslavsky, Phys. Lett. 69A, 145 (1978); P. Coullet, C. Tresser, and A. Arneodo, Phys. Lett. 77A, 327 (1980).

[8]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978), and 21, 669 (1979).

[9]P. Collet and J.-P. Eckman, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Basel, 1980).

[10]V. I. Arnold, Trans. Am. Math. Soc., 2nd Ser. 46, 213 (1965); M. R. Herman, *Geometry and Topology*, Lecture Notes in Mathematics No. 597 (Springer-Verlag, Berlin, 1977), p. 271; S. J. Shenker, to be published.

[11]T. Geisel and J. Nierwetburg, Phys. Rev. Lett. 48, 7 (1982); M. Schell, S. Fraser, and R. Kapral, to be published.

[12]R. Perez and L. Glass, to be published.

[13]S. J. Chang, M. Wortis, and J. A. Wright, Phys. Rev. A 24, 2669 (1981); S. Fraser and R. Kapral, to be published.

[14]D. G. Aronson, M. A. Chory, G. R. Hall, and R. P. McGehee, Commun. Math. Phys. 83, 303 (1982).