Exact Solutions to the Feigenbaum Renormalization-Group Equations for Intermittency

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Exact solutions to the Feigenbaum renormalization-group recursion relation, and the associated eigenvalue equations describing deterministic as well as stochastic perturbations, are found for the case of intermittency. These solutions are generated by a reformulation of the one-dimensional iterated map that exploits its topological equivalence to a translation. Direct resummation of series expansions gives the same results.

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The study of bifurcation and the transition to chaos has attracted intense interest recently, and considerable progress has been made. The three most commonly discussed scenarios,[1] associated, respectively, with the works of Feigenbaum,[2] Manneville and Pomeau,[3] and Ruelle and Takens,[4] are based on three different types of bifurcations: the pitchfork, tangent, and Hopf bifurcations. The much discussed period-doubling route to chaos is based on the pitchfork bifurcation.

The tangent bifurcation, on the other hand, offers a different route to chaos via intermittency. In this scenario, intermittency is a precursor to periodic behavior. It consists of long-lived episodes of nearly periodic behavior, the duration of which becomes arbitrarily long as the transition, via a tangent bifurcation [Fig. 2(a)], is approached.

Recently, Hirsch, Huberman, and Scalapino,[5] following the initial ideas of Manneville and Pomeau, proposed a detailed theory of intermittency. Scaling relations for the length of laminarity in the presence of noise were established[5,6] by considering a Langevin equation describing the map near the saddle point, and using the Fokker-Planck techniques to determine the time of passage. Very remarkably, Hirsch, Nauenberg, and Scalapino[7] later found that the same results can be simply explained by using the same functional renormalization-group equations first proposed by Feigenbaum in his study of period doubling—with a mere change of boundary conditions appropriate to the tangency condition. Thus the renormalization group provides a unified and elegant approach to both period doubling and intermittency.

The renormalization-group approach as formulated by Feigenbaum postulates the existence of a universal map, obtained by repeated compositions and rescalings of the original map, at the onset of chaos. The rescaling factor needed to generate the universal map yields one universal exponent. Eigenvalues describing the rate at which perturbations of this map grow provide the others.

To find the spectrum of eigenvalues and corresponding eigenfunctions, Hirsch, Nauenberg, and Scalapino used series-expansion techniques. In the simplest $z=2$ case they were able to sum the series and obtain a closed-form solution to the universal function. However, the universal function for arbitrary z and all eigenfunctions were only computed to the first few orders.

We have found that it is possible to obtain not only all the exponents for intermittency, but also closed-form results for the universal functions and all eigenfunctions corresponding to deterministic as well as stochastic perturbations for arbitrary z. This was achieved by a simple transformation that recasts the map near a tangent bifurcation into a simple translational map $x_{i+1}=x_i+b$, with b a constant. Direct resummation of series expansions corroborates our results. Whether this technique will prove to be of general utility remains to be seen, but the remarkable simplification it leads to in the renormalization-group study of intermittency induces us to believe that it may well prove a useful tool in the study of other dynamical transitions.

The tangent bifurcation as it occurs in iterated one-dimensional maps is illustrated in Fig. 1. Here the map $f(x)=rx(1-x)$ and its third iterate $f^{(3)}(x)=f(f(f(x)))$ are shown at $r=r_3=1+\sqrt{8}$. For

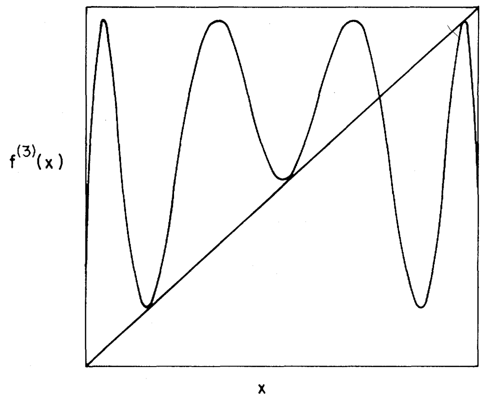

FIG. 1. The third-iterated map $f^{(3)}(x)$ at r_3.

$r \lesssim r_3$, $f^{(3)}(x)$ has two unstable fixed points, at $x=0$ and $x=(r-1)/r$. These are the unstable fixed points of $f(x)$. As r passes through r_3 [see Fig. 2(a)], $f^{(3)}(x)$ acquires six new fixed points, three stable and three unstable. The three stable fixed points are the three elements of a stable period-three limit cycle of $f(x)$. Even though the map has no stable period-three cycle when $r \lesssim r_3$, it is evident [Fig. 2(b)] that under repeated iterations of $f^{(3)}(x)$, x_i's spend several iterations in the immediate vicinity of the points of closest approach to the 45° line. This behavior corresponds to orbits under $f(x)$ that look nearly periodic for a sizable number of iterations, i.e., they almost repeat themselves every third iteration, but eventually slip out of this pattern, and, shortly thereafter, establish another pattern of near periodicity. This sequence of long-lived episodes is the phenomenon of intermittency.

To study the transition to periodicity of order n we consider the nth iterated map in the immediate vicinity of one of the n points at which it achieves tangency to the 45° line at the transition. Shifting the origin of coordinates to that point, we have for the map at tangency

$$f^n(x) = x(1+ux^{z-1}) + O(x^z), \tag{1}$$

where u is the coefficient of expansion. The exponent z determines the "universality classes." Most commonly z will be equal to 2. Here we keep it general with the understanding that

$$x^{z-1} \equiv |x|^{z-1}\operatorname{sgn}(x). \tag{2}$$

The universal map $f^*(x)$ has a power-series expansion in x whose two lowest-order terms match the right-hand side of Eq. (1). The map furthermore satisfies

$$f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x), \tag{3}$$

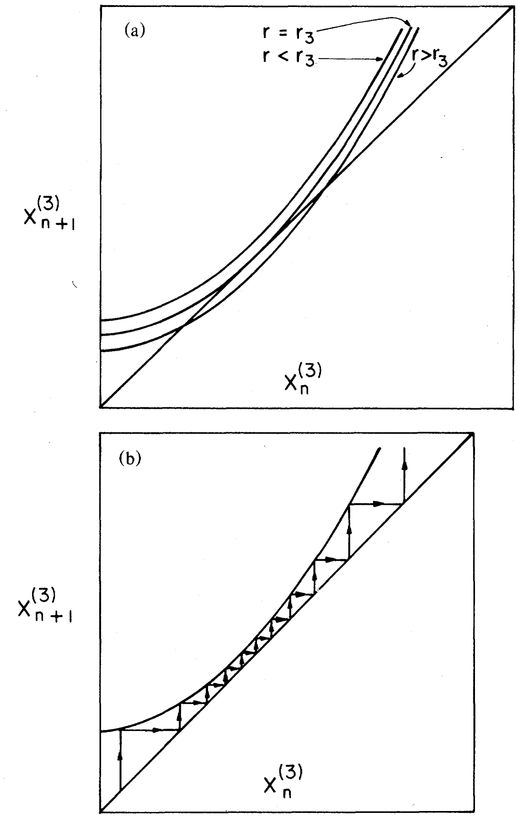

FIG. 2. (a) Tangent bifurcation near r_3. (b) Slow passage through the channel region.

where α is the rescaling factor mentioned earlier. If we add a small perturbation $\epsilon h_\lambda(x)$ to $f^*(x)$ then the composition of $f_\epsilon(x) = f^*(x) + \epsilon h_\lambda(x)$ satisfies

$$f_\epsilon(f_\epsilon(x)) = \alpha^{-1} f^*(\alpha x) + \epsilon(\lambda/\alpha) h_\lambda(\alpha x) + O(\epsilon^2) \tag{4}$$

when the eigenfunction $h_\lambda(x)$ satisfies

$$f^{*\prime}(f^*(x)) h_\lambda(x) + h_\lambda(f^*(x)) = (\lambda/\alpha) h_\lambda(\alpha x). \tag{5}$$

There will actually prove to be a *spectrum* of eigenvalues λ and corresponding eigenfunctions $h_\lambda(x)$. Stochastic exponents[8,9] are associated with the rate of growth of stochastic perturbations of the form $\xi g_{\lambda_g}(x)$, with ξ a random variable controlled by a probability distribution of unit width. Here the eigenfunctions satisfy

$$f^{*\prime 2}(f^*(x)) g_{\lambda_g}{}^2(x) + g_{\lambda_g}{}^2(f^*(x)) = (\lambda_g/\alpha)^2 g_{\lambda_g}{}^2(\alpha x). \tag{6}$$

Consider now the following recursion relation:

$$G(x') = G(x) - a \tag{7}$$

with $G(x)$ a function to be determined shortly. Iterating this recursion relation we have

$$G(x'') = G(x') - a = G(x) - 2a\,. \tag{8}$$

We can generate the universal map for intermittency by choosing a $G(x)$ for which a rescaling of x yields the original recursion relation Eq. (7) from the iterated recursion relation Eq. (8). Such a function satisfies

$$G(x) = 2G(\alpha x)\,. \tag{9}$$

The form we need is $G(x) = x^{-(z-1)}$, with

$$\alpha = 2^{1/(z-1)}\,. \tag{10}$$

Note that here the quantity z is arbitrary. Thus, a function satisfying Eq. (3) is obtained by recasting the recursion relation Eq. (7) into explicit form. Using $G(x) = x^{-(z-1)}$ we obtain

$$x' = f^*(x) = (x^{-(z-1)} - a)^{-1/(z-1)}\,. \tag{11}$$

Replacing a by $(z-1)u$ in Eq. (11) we have

$$f^*(x) = x[1-(z-1)ux^{z-1}]^{-1/(z-1)}\,. \tag{12}$$

It can be verified explicitly that $f^*(x)$ in Eq. (12) satisfies Eq. (3) and reproduces the correct power-series expansion. Furthermore the scale factor α given by Eq. (10) is correct.

We now consider the effect of a perturbation to $G(x)$ in Eq. (7). Our new implicit recursion relation is

$$x'^{-(z-1)} + \epsilon H(x') = x^{-(z-1)} + \epsilon H(x) - u(z-1)\,. \tag{13}$$

If $H(x) = x^{-p}$, then iterating the recursion relation Eq. (13) and rescaling by α as given by Eq. (10), we obtain our original recursion relation except that the coefficient ϵ has been increased by the factor λ, where

$$\lambda = 2^{p-z+1/(z-1)}\,. \tag{14}$$

The associated eigenfunction is obtained by recasting Eq. (13) into an explicit recursion relation. Solving for x' in terms of x to order ϵ we obtain

$$\begin{aligned} x' &= x[1-u(z-1)x^{z-1}]^{-1/(z-1)} - \frac{\epsilon}{z-1}[x^{-(z-1)} - u(z-1)]^{-z/(z-1)}\{x^{-p} - [x^{-(z-1)} - u(z-1)]^{p/(z-1)}\} + O(\epsilon^2) \\ &\equiv f^*(x) - \frac{\epsilon u p}{z-1} h_\lambda(x) + O(\epsilon^2)\,. \end{aligned} \tag{15}$$

The eigenfunction $h_\lambda(x)$ has been normalized so that its lowest-order term in x is x^{2z-1-p}. If we want an eigenfunction corresponding to a shift from tangency, that lowest-order term must be 1, and so we must choose $p = 2z-1$, which means than λ in Eq. (14) is equal to $2^{z/(z-1)}$. This matches with the relevant eigenvalue of Hirsch, Nauenberg, and Scalapino.[7]

The stochastic eigenfunctions are variants of the nonstochastic ones. They are

$$\begin{aligned} g_{\lambda_g}{}^2(x) &= (1/uq)[x^{-(z-1)} - u(z-1)]^{-2z/(z-1)}\{x^{-q} - [x^{-(z-1)} - u(z-1)]^{q/(z-1)}\} \\ &= (x/uq)^{2z-q}\{[1-u(z-1)x^{z-1}]^{-2z/(z-1)} - [1-u(z-1)x^{z-1}]^{-(2z-q)/(z-1)}\}\,. \end{aligned} \tag{16}$$

with

$$\lambda_g = 2^{[q-2(z-1)]/2(z-1)}\,. \tag{17}$$

The lowest-order term in $g_{\lambda_g}{}^2$ is x^{3z-1-q}. If we want that term to be a constant we must choose $q = 3z-1$, in which case λ_g in Eq. (17) is equal to $2^{(z+1)/2(z-1)}$. All these results can also be obtained directly by resumming the series expansions.

The fact that a reformulation of the recursion relation leads to an immediate and complete solution of the renormalization-group equations for the iterated map near a tangent bifurcation is highly intriguing. Whether or not this kind of reformulation proves useful in the study of other transitions in dynamical systems remains to be seen. It certainly deserves to be considered as a viable approach.

This complete set of exact solutions provides a rare laboratory where ideas and theories can be experimented with and tested. The underlying mathematical structure, physical implications, and experimental consequences are still to be ruminated. However, since the method employed here depends crucially on the fact that the map for intermittency is topologically equivalent to a translation, most likely it will not prove to be fruitful for the study of period doubling.

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